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PRODUCTIVITY DISPERSION, BETWEEN-FIRM COMPETITION, AND THE LABOR SHARE

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I study the effect of labor market imperfections on the labor share in a tractable model that emphasizes the interaction between productivity dispersion and firm competition for workers. I calibrate the model using administrative data covering the universe of firms in Canada from 2000 to 2015. As in the data, most firms have a high labor share, yet the aggregate labor share is low due to the disproportionate effect of a small fraction of large, highly productive firms. I find that a rise in the dispersion of firm productivity causes the aggregate labor share to decline in favor of firm profits. The mechanism is that productivity dispersion effectively shields high-productivity firms from wage competition. Regression evidence from cross-country and cross-industry data supports both the model prediction and mechanism.

KEYWORDS: Productivity dispersion, firm dynamics, search frictions, labor share.

1. INTRODUCTION

FOR MOST OF THE TWENTIETH CENTURY, worker compensation in the U.S. grew one-for-one with labor productivity. This observation has been enshrined as one of the stylized facts of economic growth (Kaldor (1961)). However, starting around 1990, worker compensation has stagnated relative to labor productivity, leading to a decline of the labor share (Elsby, Hobijn, and Şahin (2013); Karabarbounis and Neiman (2014)). Over that same period, there has been a sustained rise in the dispersion of firm productivity, where the labor productivity gap between the best firms and the rest has widened (Kehrig (2015); Barth, Bryson, Davis, and Freeman (2016)).

In this paper, I study the effect of labor market imperfections on the labor share in a model that emphasizes the interaction between productivity dispersion and firm competition for workers. First, I build a tractable model of firm dynamics with search frictions and wage posting in the labor market. I use the model to generate theoretical predictions regarding the link between the distribution of firm productivity and the aggregate labor share. Second, I calibrate and validate the model using administrative data covering the universe of firms in Canada over the 2000–2015 period. Third, I use the model to simulate the effect of a rise in productivity dispersion and contrast its predictions with evidence from cross-industry and cross-country data.

I have two main findings. First, I find that labor market imperfections have large and heterogeneous effects on firm-level labor shares. Through the lens of the model, labor market imperfections depress the aggregate labor share by 15 percentage points (compared to a frictionless model) yet they *raise* the average firm-level labor share by 12 percentage points. This asymmetric effect is due to the fact that value-added is concentrated within large, highly productive firms that exert significant monopsony power (i.e., they pay wages below the marginal product of labor).

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Second, I establish theoretically that a rise in productivity dispersion causes the aggregate labor share to decline in favor of firm profits. The amount of competition that firms face in the model is effectively determined by how close they are to their competitors in terms of productivity. Rising productivity dispersion thus shields high-productivity firms from wage competition. Quantitatively, I find that a rise in productivity dispersion of the same magnitude as what the U.S. economy has experienced over the 2000–2015 period causes a 1.3 percentage point decline in the aggregate labor share. Interestingly, the effect of a rise in productivity dispersion works through a reallocation of value-added toward firms with a low (and declining) labor share, rather than a broad-based decline of firm-level labor shares.

Overview of the Paper

In Section 2, I present the model and derive a number of theoretical predictions. The model embeds a labor market with search frictions, wage posting, and on-the-job search (as in Burdett and Mortensen (1998)) in a continuous-time firm dynamics model where firms face productivity shocks (as in Coles and Mortensen (2016)). In the model, firms solve a dynamic problem and choose how much capital to rent, what wage to offer, and when to exit. In equilibrium, high-productivity firms offer high wages in order to poach workers from lower-paying firms and, therefore, *grow* faster. Relative to existing wage-posting models, my environment includes capital as a factor of production as well as endogenous firm entry and exit. These ingredients are key: introducing capital allows the model to generate an empirically reasonable level for the labor share and free entry ensures that ex ante rents are exhausted.

I obtain an analytical solution for the stationary equilibirum of the model, which allows me to generate a number of new theoretical predictions. At the micro level, I show that firm-level labor shares are (asymptotically) decreasing in firm productivity.[1] The reason is that the pass-through of productivity to wages is proportional to the (employment-weighted) density of firms. Firms in the right tail of the productivity distribution have few direct competitors and are therefore able to recruit and retain workers despite paying wages below the marginal product of labor. In contrast, the least productive firms pay wages that are *above* their marginal product and, therefore, have a high labor share. For these firms, the option value of retaining their workers compensates for the negative flow profits.

The main theoretical contribution of the paper is to show that the aggregate labor share is decreasing in the level of productivity dispersion (i.e., the "thickness" of the right tail of the productivity distribution). To establish this result, I focus on a special case where the productivity distribution is Pareto and the discount rate is zero. I show that, all else equal, the aggregate labor share is decreasing in productivity dispersion. The intuition is that when firms in the right tail of the productivity distribution become "further apart" in terms of productivity, they effectively face less competition, and thus earn higher monopsony profits. I also show that a key determinant of the sensitivity of the labor share to a change in productivity dispersion is the *persistence* of productivity differentials across firms. A more persistent productivity process implies a larger labor share decline in response to a rise in productivity dispersion. Intuitively, what matters is not only how large productivity differentials across firms are, but also how persistent they are.

[1]The precise statement is that there exists a productivity threshold after which the labor share of a firm is decreasing in its productivity.

In Section 3, I calibrate and validate the model using administrative data covering the universe of corporations in Canada over the 2000–2015 period. The data set is ideal as it allows me to compute labor productivity, labor share, capital stock, and employment at the firm level. I calibrate the model by targeting moments related to the firm productivity process as well as the pace of job reallocation between firms. Despite being very parsimonious, the model matches a number of nontargeted (within-industry) moments. For example, the aggregate labor share is 0.62 (0.65 in the data) while the average (unweighted) firm labor share is 0.89 (0.88 in the data). The reason why the average labor share is so high is that the bottom 3 deciles of firms in terms of labor productivity have labor share above one (bottom two deciles in the data). The model also predicts that value-added is concentrated within high-productivity, low-labor-share firms: the top decile of firms in terms of labor productivity account for 55% of output (42% in the data) and have a labor share of 0.53 (0.44 in the data). Finally, I assess the ability of the model to match "panel moments" such as the pass-through of labor productivity to wages and find that the model provides a satisfactory fit.

In Section 4.1, I use the calibrated model to simulate the effect of a rise in productivity dispersion. To motivate the model experiment, I show that, in recent decades, there has been a divergence between the U.S. and Canadian economies. Both productivity dispersion and the labor share have been stable in Canada, while the U.S. has experienced a rise in productivity dispersion coupled with a decline of the labor share.

The model experiment consists of simulating the effect of a rise in productivity dispersion of the same magnitude as what the U.S. experienced over the 2000–2015 period. The model predicts a labor share decline of 1.3 percentage points. Using a shift-share decomposition, I show that the decline of the labor share in the model is mostly driven by a reallocation of value-added shares between firms, rather than a broad-based decline of firm-level labor shares. High-productivity firms experience an increase in their value-added share as well as a decrease in their labor share. In contrast, low-productivity firms experience a decrease in their value-added share combined with an *increase* in their labor share. Through the lens of the model, roughly one quarter of the divergence between the Canadian and U.S. labor shares over the 2000–2015 period can be explained by differential trends in productivity dispersion.

The micro predictions of the model are in line with what has been documented by Autor, Dorn, Katz, Patterson, and Van Reenen (2020) and Kehrig and Vincent (2021) using U.S. firm-level data. In particular, the model predicts that in response to an increase in productivity dispersion: (1) the aggregate labor share declines, (2) the decline is driven by a reallocation of value-added toward firms with a low (and declining) labor share, and (3) sales concentration increases.

In Section 4.2, I compare the predictions of the model with regression evidence from cross-country and industry data. First, I use industry data from 69 Canadian 3-digit industries over the 2000–2015 period. I estimate cross-sectional, fixed effects, and long-differences regression models. In all cases, I find that industries with high (rising) productivity dispersion have a low (declining) labor share. The results are robust to controlling for other factors such concentration (as in Barkai (2020), Autor et al. (2020)) and the capital-output ratio (as in Karabarbounis and Neiman (2014)). I also test the model-implied mechanism by estimating the effect of productivity dispersion along the firm productivity distribution. As predicted by the model, a productivity dispersion shock has an asymmetric effect on firm labor shares and leads to a reallocation of value-added toward high-productivity firms. Second, I collect harmonized data on productivity dispersion and the labor share for 7 OECD countries over the 2001–2011 period and find quantitatively similar results.

Finally, in Section 5, I consider a number of model extensions (including the case with endogenous vacancy creation) in order to assess how sensitive the results of the model experiment are to different modeling choices. Overall, I find that the baseline model is somewhat conservative, in the sense that most of the model extensions considered imply a larger labor share decline in response to a rise in productivity dispersion.

The Appendix to this paper (available in Online Supplementary Material (Gouin-Bonenfant (2022)) contains all proofs and derivations as well as model extensions and robustness checks.

Related Literature

The main contribution of this paper is to link the rise in productivity dispersion to the decline of the labor share, both theoretically and empirically.[2] My theoretical results are related to a prior finding in Mangin (2017), who studies a search model where heterogeneous vacancies compete for unemployed workers via a second-price auction. Under some conditions, the author shows that the aggregate labor share is decreasing in the tail index (i.e., the inverse of the Pareto exponent) of the match-specific productivity distribution. Relative to this paper, my theoretical result emphasizes the role of firm dynamics. In particular, I show that what determines the aggregate labor share is not only the dispersion of productivity across firms, but also the persistence of these productivity differentials, as well as the pace of job reallocation between firms. In addition, I bring my model to rich firm-level data, which allows me to validate the model and then simulate the quantitative effect of a rise in productivity dispersion.

Another related finding emerges in Hartman-Glaser, Lustig, and Xiaolan (2019), who study the link between the volatility of sales and the capital share in a firm dynamics model where risk-neutral firms insure risk-averse workers. Through the lens of their model, they find that an increase in firm-level risk generates an increase in the aggregate capital share. My mechanism differs in that productivity dispersion reduces the effective amount of competition faced by high-productivity firms, while they emphasize the insurance contract between workers and shareholders.

My findings complement a growing literature that studies the importance of firm heterogeneity in shaping the decline of the U.S. labor share. Kehrig and Vincent (2021) focus on the manufacturing sector over the 1967–2012 period. They find that the decline of the labor share was driven by a reallocation of value-added toward firms with a low (and declining) labor share, but that there was limited reallocation of inputs (labor and capital). Autor et al. (2020) use data covering most sectors of the U.S. economy over the 1982–2012 period. They also find that the labor share decline was mostly driven by reallocation of market shares between firms, as opposed to a broad-based decline of labor shares within firms. In addition, they also show that industries that saw the largest increases in sales concentration also saw the largest declines in their labor shares. I contribute to this literature by providing (i) new evidence on the relationship between labor share and labor productivity at the firm-level, (ii) a theory that can explain the large heterogeneity in labor shares across firms, and (iii) a new mechanism that generates a "reallocation-driven" labor share decline.

[2]Several studies document a decline of the labor share in the U.S. and other countries (see, e.g., Elsby, Hobijn, and Şahin (2013), Karabarbounis and Neiman (2014)). A less appreciated fact is that productivity dispersion has increased over time. Evidence for the U.S. include Kehrig (2015), Barth et al. (2016) and Decker, Haltiwanger, Jarmin, and Miranda (2018). Evidence for OECD countries include Andrews et al. (2016), and Berlingieri, Blanchenay, and Criscuolo (2017).

My paper contributes to the literature on the labor share by studying the aggregate effects of search frictions. Jarosch, Nimczik, and Sorkin (2019) study a "granular" search model, where wages are determined by Nash bargaining. The main difference with my paper is that they study market power as arising due to firm size, while in my model there is no market power in the traditional sense (firms are atomistic).[3] Instead, I focus on the rent-sharing problem between workers and firms, and in particular how productivity dispersion affects the optimal wage-setting decision of firms.

Relative to the frictional monopsony literature—which studies how monopsony power arises in the presence of search frictions and wage posting—my main contribution is to establish a theoretical link between productivity dispersion and the aggregate labor share and to quantify its importance.[4] The most closely related papers are Bontemps, Robin, and Van den Berg (2000) and Cahuc, Postel-Vinay, and Robin (2006). Both papers structurally estimate search models using French administrative data on wages and firm productivity. Cahuc, Postel-Vinay, and Robin (2006) quantifies the contributions of bargaining power and between-firm competition in raising wages above the value of unemployment. They find that between-firm competition is the most important factor. Bontemps, Robin, and Van den Berg (2000) use a wage-posting model and find that high-productivity firms exert more monopsony power, which is closely related to my finding that high-productivity firms have a lower labor share. Compared to these papers, who focus on the distribution of wages, my paper focuses on the *aggregate* labor share, and in particular how productivity dispersion affects the aggregate profit share.

My model differs from most existing wage-posting models in that it incorporates firm dynamics (endogenous entry, exit, and growth). Existing firm dynamics models with search frictions include Elsby and Michaels (2013), Kaas and Kircher (2015), Coles and Mortensen (2016), Gavazza, Mongey, and Violante (2018), Elsby and Gottfries (2021), Bilal, Engbom, Mongey, and Violante (2019). My model is most closely related to Coles and Mortensen (2016) (henceforth Coles–Mortensen).

In terms of the environment, my model differs from Coles–Mortensen along three main dimensions. First, I add capital as a factor of production. Payments to capital affect the share of value-added that accrues to workers. Second, I model the entry and exit decision of firms, while Coles–Mortensen assumes that firm productivity is high enough so that firms always make positive profits and, therefore, always wish to enter and never wish to exit. Free entry ensures that ex ante rents are exhausted (i.e., the marginal entrant makes zero profits in expectation), thereby limiting the amount of profits in equilibrium. Moreover, endogenous exit allows the model to rationalize the existence of temporary negative profits (an important feature of the data). Finally, Coles–Mortensen model the hiring process using a hiring cost function. Instead, I assume "balanced matching" as in Burdett and Vishwanath (1988) (i.e., exogenous meeting rates that are proportional to firm size),

[3]Several recent papers study the aggregate implications of market power in frictionless models. For instance, see Barkai (2020), Eggertsson, Robbins, and Wold (2021), De Loecker, Eeckhout, and Unger (2020), and Edmond, Midrigan, and Xu (2018) for product market power and Azar, Marinescu, and Steinbaum (2020), Benmelech, Bergman, and Kim (2020), Berger, Herkenhoff, and Mongey (2019), and Brooks, Kaboski, Li, and Qian (2021) for labor market power.

[4]Recent applications of the Burdett and Mortensen (1998) model to study the wage-setting behavior of firms include: Meghir, Narita, and Robin (2015), who study the role of informal labor markets, Engbom and Moser (2021), who study the effect the minimum wage on inequality in Brazil, Heise and Porzio (2021) who study the role of firms and location preferences in shaping spatial wage gaps, and Bilal and Lhuillier (2021), who study the effect of outsourcing.

which delivers analytical tractability and allows me to obtain my main theoretical results.[5] Finally, in Section 5.3, I relax the balanced matching assumption by considering a model extension with endogenous vacancy-posting.

2. THEORY

I now present the baseline model. I start by describing the environment and assumptions, paying particular attention to dimensions in which it differs from Burdett and Mortensen (1998) and Coles and Mortensen (2016). The key theoretical results are in Section 2.3, where I obtain model predictions regarding the effect of productivity dispersion on the aggregate labor share.

2.1. *Environment*

The economy is populated by an endogenous measure F of heterogeneous firms and a unit measure of identical, infinitely-lived workers. Firms and workers are risk neutral and discount the future using an exogenous discount rate $r > 0$. Time is continuous.

Technology

Firms compete for workers by posting wage contracts w in a frictional labor market and rent capital in a perfectly competitive market at user cost $R \equiv r + d$, where $d \geq 0$ is the depreciation rate of capital. Firms can change the wage policy at any time and cannot commit to future wages. I assume that firms are required to pay all of their workers the same wage. Workers can be either employed or unemployed and are available to receive job offers in both states. The flow value of unemployment is exogenous and given by $b > 0$.

Firm output Y is given by a constant return to scale Cobb–Douglas production function

$$Y = zK^{\alpha}N^{1-\alpha}, \tag{1}$$

where z is productivity, $\alpha \in [0, 1)$ is the capital share, $N \geq 0$ is the measure of workers that the firm employs, and $K \geq 0$ is the capital stock.

Firm-level productivity z obeys a jump process. At Poisson rate $\chi > 0$, continuing firms draw a new productivity level from an exogenous distribution Γ_0.[6] Firm productivity thus remains constant for a stochastic duration and then jumps to a new level that is independent from the previous one. I assume that Γ_0 satisfies the following conditions.

[5]In particular, I show that the stationary equilibrium of the model is the solution to a system of quadratic equations (Proposition 3). The resulting tractability of the model allows me to establish my main theoretical result (Proposition 6). In terms of new theoretical results, I also provide a sufficient condition for the existence of a stationary equilibrium (Assumption 2) and a characterization of the Pareto exponent of the firm-size distribution (Appendix Proposition 9). I use theoretical results on power laws (i.e., Toda (2014); Beare, Seo, and Toda (2021); Beare and Toda (2022)) and the "Pareto extrapolation" solution method (developed in Gouin-Bonenfant and Toda (2022)) to solve the firm size distribution. The emergence of fat-tailed size distribution in my model echoes findings in Luttmer (2007) and Luttmer (2011), who show that homogeneous firm dynamics models naturally give rise to fat-tailed distributions.

[6]A Poisson rate $\chi > 0$ means that the time elapsed between two events (in this case a productivity reset) is a random variable that is exponentially distribution with mean $1/\chi$.

ASSUMPTION 1: *The productivity distribution* $\Gamma_0 : \mathbb{R}_+ \to [0, 1]$ *is differentiable and satisfies*

$$\Gamma_0'(z) > 0, \qquad \int_0^\infty z\Gamma_0'(z)\,\mathrm{d}z = 1, \qquad \int_0^\infty z^{\frac{1}{1-\alpha}}\Gamma_0'(z)\,\mathrm{d}z < \infty.$$

The first condition ensures that the density is strictly positive everywhere, the second condition is a normalization, and the third condition ensures that aggregate output is finite. For example, the third condition rules out Pareto distributions with exponents smaller than $\frac{1}{1-\alpha}$.

Firm Dynamics and Matching

At Poisson rate $\mu > 0$, an unemployed worker meets a potential firm. The potential firm then draws a productivity level $z \sim \Gamma_0$ and decides whether or not to enter. Continuing firms meet new workers at rate λN, where the meeting rate $\lambda > 0$ is a technological parameter and N is the measure of workers currently employed. This assumption represents a departure form the canonical Burdett and Mortensen (1998) model, who assume that firms meet workers at a rate that is independent of size. Burdett and Vishwanath (1988) call this linear meeting rate "balanced matching." (In Section 5.3, I extend the model to allow for endogenous vacancy-posting and show that balanced matching arises a limiting case.)

Matching is random, which implies that a fraction u of meetings are with unemployed workers while the remainder $1 - u$ are with employed workers, where u is the unemployment rate. When a firm meets a worker, the worker only observes the wage posted by the firm and decides whether or not to accept the job offer. At Poisson rate $\delta \geq 0$, continuing workers exogenously separate to unemployment. At any moment, a firm can choose to exit and send all of its workers to unemployment.

Together, the assumptions of (i) constant return to scale in production, (ii) balanced matching, and (iii) the jump process for firm productivity allow me to obtain an analytical characterization of the stationary equilibrium of the model. As we will see shortly, the reason is that the firm problem becomes homogeneous (firm size is not a payoff relevant state variable) and that the stationary distribution is the solution to a quadratic equation.

Strategies and Beliefs

When an employed worker receives an offer from a competing firm, she must form expectations about the future path of wages at both firms and choose the alternative with the highest continuation value. To make the analysis tractable, I restrict attention to Markov strategies that depend only on productivity. Denote by $w(z)$ the wage offered by a firm with productivity z, $e(z) \in \{0, 1\}$ the entry decision of a potential firm, and $x(z) \in \{0, 1\}$ the exit decision of a continuing firm. As in Coles and Mortensen (2016), I focus on equilibria where firm strategies do not depend directly on size. This is a natural choice given that the firm problem is homogeneous.

Distributions

Two related distributions will play a central role in my analysis: the employment-weighted productivity distribution $P(z)$ and the wage distribution $\widetilde{P}(w)$. The employment-weighted productivity distribution summarizes the allocation of workers to firms of different productivity levels. Its definition is that, for a given productivity level z, a share

$P(z) \in [0,1]$ of workers are at firms with productivity less than z. When more productive firms pay strictly higher wages (i.e., $w'(z) > 0$)—as will be the case in equilibrium—the following relationship holds:

$$P(z) = \widetilde{P}\big(w(z)\big). \tag{2}$$

In words, it says that the share of workers at firms with productivity less than z is equal to the share of workers earning a wage below $w(z)$.

Worker Problem

Denote by U the value of being unemployed and by $W(w)$ the value of being employed at a firm currently paying wage a w. The value functions are the solutions to standard Bellman equations:

$$rU = b + \underbrace{\mu \int e(z)\big|W\big(w(z)\big) - U\big|_+ \,\mathrm{d}\Gamma_0(z)}_{\text{job offers by entering firm}} + \underbrace{\lambda(1-u)\int \big|W\big(w(z)\big) - U\big|_+ \,\mathrm{d}P(z)}_{\text{job offers by continuing firm}}, \tag{3}$$

where $|x|_+ \equiv \max\{x, 0\}$, for any variable x. The first term on the right-hand side of (3) accounts for the flow value of unemployment b while the second and third terms account for job offers by entering and continuing firms.

$$\begin{aligned} rW(w) = w &+ \underbrace{\chi \int \big(1 - x(z)\big)\big(W\big(w(z')\big) - W(w)\big)\,\mathrm{d}\Gamma_0(z')}_{\text{wage changes due to productivity shocks}} \\ &+ \underbrace{\lambda(1-u)\int \big|W\big(w(z')\big) - W(w)\big|_+ \,\mathrm{d}P(z')}_{\text{job offers by continuing firm}} \\ &+ \underbrace{\left(\chi \int x(z)\,\mathrm{d}\Gamma_0(z) + \delta\right)\big(U - W(w)\big)}_{\text{job destruction}}. \end{aligned} \tag{4}$$

The first term on the right-hand side of (4) accounts for the flow of wages, the second term accounts for wage changes due to firm productivity shocks, the third term accounts for job offers by continuing firms, and the fourth term accounts for job destruction. Given that employed workers can quit to unemployment, the value of being employed must be weakly greater than the value of being unemployed:

$$W(w) \geq U. \tag{5}$$

Proposition 1 characterizes the best response of workers to firms' decision rules $w(z)$, $e(z)$, and $x(z)$ along an equilibrium path.

PROPOSITION 1: *Unemployed workers accept a job offer if and only if it exceeds a reservation wage $\underline{w}$ given by*

$$\underline{w} = b + \int \big(\mu e(z) - \chi\big(1 - x(z)\big)\big)\big(W\big(w(z)\big) - U\big)\,\mathrm{d}\Gamma_0(z). \tag{6}$$

Employed workers accept a job offer if and only if it exceeds their current wage. They quit to unemployment if and only if their current wage falls below $\underline{w}$.

Firm Growth

Equipped with the workers' decision rule, I now characterize the link between firm-level wage and employment growth. The instantaneous change in employment at a firm of size N paying $w \geq \underline{w}$ is given by

$$\mathrm{d}N_t = \widetilde{g}(w_t)N_t\,\mathrm{d}t, \tag{7}$$

where the employment growth function of continuing firms $\widetilde{g}(w)$ is an endogenous object that depends on the wage distribution $\widetilde{P}(w)$. A simple expression can be obtained for the employment growth function.

LEMMA 1: *For all* $w \geq \underline{w}$, *the employment growth function is given by*

$$\widetilde{g}(w) \equiv \underbrace{\lambda u + \lambda(1-u)\widetilde{P}(w)}_{\textit{hiring rate}} - \underbrace{\lambda(1-u)\big(1-\widetilde{P}(w)\big) - \delta}_{\textit{separation rate}}. \tag{8}$$

The function $\widetilde{g}(\cdot)$ *is weakly increasing and bounded from above by* $\lambda - \delta$.

Lemma 1 highlights two important predictions of the model. First, high-wage firms grow faster. The reason is that offering a higher wage increases net poaching flows (i.e., the net amount of workers poached from other firms). Second, the marginal benefit of increasing the wage eventually converges to zero. To understand why, consider the case of a firm that offers the highest wage in the economy. Note that this firm is able to hire every worker that it meets and loses none of its workers to other firms. Hence, increasing the wage further would not lead to a higher growth rate of employment.

ASSUMPTION 2: *The exogenous rates* $(\lambda, \chi, \mu, \delta)$ *satisfy*

$$0 < \lambda - \delta < \min\{\chi, \mu\}$$

Assumption 2 is a sufficient condition for the existence of a steady-state unemployment rate and a well-defined employment-weighted productivity distribution (more on that in Section 2.2). The intuition is that if $\lambda - \delta \geq \min\{\chi, \mu\}$, some firms would grow too fast for too long. As a result, the stationary firm size distribution would have an infinite mean.

Firm Problem

At every instant, the problem of a firm is to choose the current wage level w, the amount of capital per worker k, and whether to exit or continue operating. Given the constant return to scale assumption in both production and matching, the firm problem is homogeneous. Without loss of generality, I will characterize the value of a firm with a unit measure of employment $v(z) \equiv v(z, 1)$.

In the productivity region where the value of the firm is positive, the value function is the solution to the following *Hamilton–Jacobi–Bellman* (HJB) equation:

$$rv(z) = \max_{w\geq b, k\geq 0}\big\{\underbrace{zk^{\alpha} - w - Rk}_{\text{profits}} + \underbrace{v(z)\widetilde{g}(w)}_{\text{growth}}\big\} + \underbrace{\chi\left(\int v(x)\,\mathrm{d}\Gamma_0(x) - v(z)\right)}_{\text{productivity shocks}}. \tag{9}$$

In the productivity region where the value of the firm is negative, the firm decides to exit.[7]

The first-order conditions for capital and wages are respectively given by

$$\underbrace{z\alpha k^{\alpha-1}}_{\substack{\text{marginal product}\\ \text{of capital}}} = \underbrace{R}_{\substack{\text{user cost}\\ \text{of capital}}}, \tag{13}$$

$$\underbrace{1}_{\substack{\text{marginal increase}\\ \text{in the wage bill}}} = \underbrace{v(z)\widetilde{g}'(w)}_{\substack{\text{marginal increase in}\\ \text{the value of the firm}}}. \tag{14}$$

The first-order condition for capital is standard and implies that firms rent capital up to the point where the marginal product of capital is equal to its user cost. The first-order condition for wages captures the trade-off between current profits and future growth. When a firm offers a high wage, it makes lower per-employee profit but grows faster, which increases the present value of future profits. The following proposition provides closed-form expressions for the policy functions.

PROPOSITION 2: *Capital per worker $k(z)$, the wage schedule $w(z)$, the entry decision $e(z)$, and the exit decision $x(z)$ are given by*

$$k(z) = \alpha^{\frac{1}{1-\alpha}} R^{-\frac{1}{1-\alpha}} z^{\frac{1}{1-\alpha}}, \tag{15}$$

$$w(z) = \underline{w} + \int_{\underline{z}}^{z} v(x) g'(x)\, \mathrm{d}x, \tag{16}$$

$$e(z) = 1\{z \geq \underline{z}\}, \qquad x(z) = 1\{z < \underline{z}\}, \tag{17}$$

where $\underline{z} \geq 0$ is a threshold. Capital per worker and wages are increasing in productivity z.

Denote the equilibrium growth rate of employment by $g(z) \equiv \widetilde{g}(w(z))$. Combining the expression for $\widetilde{g}(w)$ (see equation (8) and the definition of the employment-weighted productivity distribution $P(z)$ in the case where $w'(z) > 0$ (see equation (2)), I obtain

$$g(z) = \lambda u + \lambda(1-u)P(z) - \lambda(1-u)\bigl(1 - P(z)\bigr) - \delta. \tag{18}$$

Labor productivity $y(z)$ is defined as valued-added per worker while the labor share $\mathrm{LS}(z)$ is defined as payroll over value-added:

$$y(z) \equiv zk(z)^{\alpha}, \qquad \mathrm{LS}(z) = w(z)/y(z). \tag{19}$$

[7] More generally, the value function is the solution to a linear complementarity problem:

$$rv(z) \geq \max_{w \geq b, k \geq 0} \{zk^{\alpha} - w - Rk + v(z)\widetilde{g}(w)\} + \chi\biggl(\int v(x)\, \mathrm{d}\Gamma_0(x) - v(z)\biggr), \tag{10}$$

$$rv(z) \geq 0, \tag{11}$$

$$0 = v(z)\biggl(rv(z) - \max_{w \geq b, k \geq 0} \{zk^{\alpha} - w - Rk + v(z)\widetilde{g}(w)\} - \chi\biggl(\int v(x)\, \mathrm{d}\Gamma_0(x) - v(z)\biggr)\biggr). \tag{12}$$

The inequalities (10) and (11) ensure that the annuity value of a firm $rv(z)$ is never less than the continuation value or the exit value. Equation (12) requires that either the value of a firm is zero, or that its annuity value is equal to its continuation value. Alternatively, the value function can be characterized as the solution to a variational inequality of the obstacle type (see Achdou, Buera, Lasry, Lions, and Moll (2014)).

2.2. *Equilibrium*

I now define and solve the equilibrium of the model. To simplify the notation, I define the flow of entering firms μ_e, the firm exit rate χ_x, the rate of arrival of productivity shocks χ_s, and the productivity distribution of active firms $\Gamma(z)$ as

$$\begin{aligned} &\mu_e \equiv \mu\big(1-\Gamma_0(\underline{z})\big), \qquad \chi_x \equiv \chi\Gamma_0(\underline{z}), \\ &\chi_s \equiv \chi\big(1-\Gamma_0(\underline{z})\big), \qquad \Gamma(z) \equiv \frac{\Gamma_0(z)-\Gamma_0(\underline{z})}{1-\Gamma_0(\underline{z})}. \end{aligned} \tag{20}$$

In Appendix A.4, I provide a derivation of the following laws of motion for the unemployment rate u and the employment-weighted distribution $P(z)$:

$$\dot{u} = \underbrace{(\delta+\chi_x)(1-u)}_{\text{unemployment inflows}} - \underbrace{u\big(\mu_e+\lambda(1-u)\big)}_{\text{unemployment outflows}}, \tag{21}$$

$$\begin{aligned} \dot{P}(z) = {} & \underbrace{(1-u)\lambda P(z)\big(P(z)-1\big)}_{\text{job-to-job flows}} + \underbrace{\frac{u}{1-u}\mu_e\Gamma(z) + u\lambda P(z)}_{\text{employment inflows}} \\ & - \underbrace{(\delta+\chi_x)P(z)}_{\substack{\text{employment}\\ \text{outflows}}} + \underbrace{\chi_s\big(\Gamma(z)-P(z)\big)}_{\text{productivity shocks}}. \end{aligned} \tag{22}$$

Equilibrium Definition

A stationary equilibrium consists of a productivity distribution $\Gamma(z)$, rates (μ_e, χ_s, χ_x), value functions for the worker $(U, W(w))$, a reservation wage $\underline{w}$, a value function for the firm $v(z)$, a productivity threshold $\underline{z}$, decision rules for the firm $(w(z), k(z))$, an unemployment rate u, and an employment-weighted productivity distribution $P(z)$ that satisfy: (i) the truncated productivity distribution $\Gamma(z)$ and rates μ_e, χ_s, χ_x are determined by the optimal entry and exit of firms (20), (ii) the value functions for the worker U, $W(w)$ solve (3), (4), (iii) the reservation wage $\underline{w}$ solves the worker problem (6), (iv) the value function $v(z)$ and productivity threshold $\underline{z}$ solve the linear complementarity problem (10), (11), (12), (v) the decision rules $k(z)$ and $w(z)$ solve the firm problem (15), (16), and (vi) the unemployment rate u and the employment-weighted productivity distribution $P(z)$ are the stationary solutions to their respective laws of motion (21), (22).

Solution

The following proposition provides an expression for the unemployment rate u and the employment-weighted productivity distribution $P(z)$ in a stationary equilibrium.

PROPOSITION 3: *There exist a unique pair of unemployment rate u and employment-weighted distribution $P(z)$ that are stationary solutions to their respective laws of motion* (*equations* (21) *and* (22)):

$$u = \frac{\lambda+\delta+\chi_x+\mu_e-\sqrt{(\lambda+\delta+\chi_x+\mu_e)^2-4\lambda(\chi_x+\delta)}}{2\lambda}, \tag{23}$$

$$P(z) = \big(\lambda(1-u)+\chi+\delta-u\lambda$$

$$- \sqrt{\left(\lambda(1-u)+\chi+\delta-\lambda u\right)^2 - 4\lambda(1-u)(\chi+\delta-u\lambda)\Gamma(z)\right)}$$
$$/\left(2\lambda(1-u)\right). \tag{24}$$

The fact that wages are increasing in firm productivity (see Proposition 2) implies that the equilibrium exhibits *rank preservation*: the ranking of wages across firms is the same as the ranking of productivity. As in Burdett and Mortensen (1998) and Moscarini and Postel-Vinay (2013), it is the rank-preserving property of the model that yields tractability.

A corollary of Proposition 3 is that the allocation of workers to firm productivity ranks is invariant to the underlying productivity distribution $\Gamma(\cdot)$. For example, the share of workers working at firms with productivity below z' depends only on $(\lambda, \chi, \delta, u)$ and the firm rank $\Gamma(z') \in [0, 1]$, but not on the absolute productivity level z'. The feature of the equilibrium that generates this result is that the optimal quit decision of worker is ordinal; they accept any job offer that generates a wage increase.

ASSUMPTION 3: *The value of unemployment b is high enough so that* $v(\underline{z}) = 0$.

From now on, I will focus on the case where the value of unemployment b is "high enough." For low values of b, it could be that firms with $z = 0$ can profitably operate (i.e., $v(0) > 0$), which would imply that there is no firm exit. Assumption 3 contrasts with Coles and Mortensen (2016), who assume that the lower bound of the productivity distribution is such that all firms make positive profits in every date and state, which implies $v(\underline{z}) > 0$.

2.3. *Micro and Macro Predictions*

I now derive predictions regarding the link between (i) productivity and labor share at the firm level and (ii) productivity dispersion and the aggregate labor share. Before doing so, I provide intuition regarding the equilibrium relationship between wages and productivity at the firm level.

Intuition

The first-order condition for capital (13) implies that the share of payments to capital in value-added $Rk(z)/y(z)$ is equal to the output elasticity of capital α. The equilibrium wage $w(z)$ thus determines how the remaining $1-\alpha$ share of value-added is split between wages and profits. The first-order condition for wages (14) evaluated at the equilibrium wage yields the following equation for the pass-through of productivity to wages:

$$w'(z) = \underbrace{g'(z)}_{\text{growth effect}} \times \underbrace{v(z)}_{\text{value effect}}. \tag{25}$$

When a firm's productivity increases, the marginal increase in the wage offered is exactly equal to the marginal increase in the value of the firm. The value of the firm increases because (i) higher wages imply a higher growth rate (the growth effect) and (ii) the firm extracts rents from every additional worker (the value effect). Using equation (18), the growth effect can be further decomposed into two terms

$$g'(z) = \underbrace{2\lambda(1-u)}_{\text{search frictions}} \times \underbrace{P'(z)}_{\text{local competition}}. \tag{26}$$

First, search frictions mitigate the effect of wages on net poaching flows. For instance, the growth effect is lower if the meeting rate λ is low or the unemployment rate u is high. Second, the extent to which a wage increase yields higher net poaching flows depends on the *density* of workers working at firms with productivity z. At the margin, a wage increase only affects the quit behavior of workers who were initially indifferent between staying or quitting. Since $P'(z)$ is a density, it must be that $P'(z) \to 0$, so firms in the upper tail of the productivity distribution face vanishing wage competition and, therefore, have no incentive to pass productivity gains to their workers.

The fact that high-productivity firms tend to make higher profits has been discussed in Bontemps, Robin, and Van den Berg (2000) in the context of the standard Burdett–Mortensen model. A unique prediction of my model—which arises due to the interaction between productivity shocks and the endogenous exit decision—is that low-productivity firms pay wages that are *above* the static marginal product of labor. Rearranging the Bellman equation (9), the value to the firm of a worker can be decomposed into the flow profits and an option value

$$v(z) = \frac{1}{r + \chi_x - g(z)} \Bigg[\underbrace{(1-\alpha)y(z) - w(z)}_{\text{flow profits}} + \underbrace{\chi_s \bigg(\int v(x)\,\mathrm{d}\Gamma(x) - v(z) \bigg)}_{\text{option value}} \Bigg]. \tag{27}$$

Evaluating at $z = \underline{z}$ and using the fact that $v(\underline{z}) = 0$ (Assumption 3), I obtain an expression for the equilibrium wage of the lowest productivity firm:

$$\underline{w} = (1-\alpha)y(\underline{z}) + \chi_s \int_0^\infty v(x)\,\mathrm{d}\Gamma(x). \tag{28}$$

Notice that workers at firms with $z = \underline{z}$ extracts all the surplus from the match, including the option value of future productivity increases. As a result, they earn a wage that exceeds their static marginal product.

Firm-Level Labor Shares

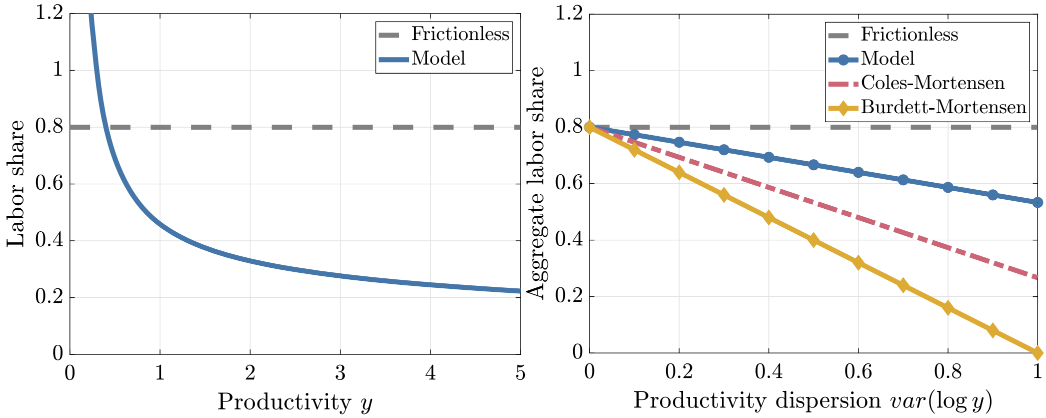
The following proposition characterizes the relationship between productivity and labor share at the firm level.

PROPOSITION 4—Micro: *There exists a threshold z' such that firms with $z < z'$ have a labor share above the frictionless benchmark* (*i.e.*, $\mathrm{LS}(z) > 1-\alpha$). *There exists a second threshold z'' such that for firms with $z > z''$, the labor share is decreasing in productivity* (*i.e.*, $\mathrm{LS}'(z) < 0$).

Figure 1a presents the relationship between the labor share and labor productivity in a numerical example. In contrast with the frictionless model where the labor share would be $1-\alpha$ for every firm, the model predicts that the labor share of high-productivity firms is lower than $1-\alpha$, while the opposite is true for low-productivity firms.

Aggregation

The aggregate labor share is equal to the output-weighted average of firm-level labor shares. I now use the analytical solution of the stationary equilibrium to obtain an expression for the aggregate labor share.



(a) Micro (b) Macro

FIGURE 1.—Relationship between labor share and productivity (numerical example).

PROPOSITION 5—Aggregation: *The aggregate labor share is given by*

$$\mathrm{LS} = \int_{\underline{z}}^{\infty} \underbrace{\mathrm{LS}(z)}_{\text{labor share}} \times \underbrace{\overbrace{\frac{\chi - \overline{g}}{\chi - g(z)}}^{\substack{\text{employment}\\ \text{share}}} \times \overbrace{\frac{y(z)}{\mathrm{Y}}}^{\substack{\text{relative}\\ \text{productivity}}}}_{\text{output share}} \times \underbrace{\mathrm{d}\Gamma(z)}_{\substack{\text{productivity}\\ \text{distribution}}}, \tag{29}$$

where $\overline{g} \equiv \int_{\underline{z}}^{\infty} g(z)\,\mathrm{d}P(z)$ *is the average growth rate of continuing firms and* $\mathrm{Y} \equiv \int_{\underline{z}}^{\infty} y(z)\,\mathrm{d}P(z)$ *is aggregate output.*

Notice that output shares are increasing in productivity z.[8] The reason is twofold. First, high-productivity firms tend to employ more workers (i.e., their employment share is higher). Second, they produce more output per worker (i.e., their relative productivity is higher). The aggregate labor share is thus disproportionately determined by the labor share of high-productivity firms.

To study the link between the *distribution* of productivity $\Gamma(z)$ and the aggregate labor share, I now restrict attention to a tractable special case that yields a closed-form solution. In particular, I assume that the discount rate r is equal to zero and that the distribution of productivity is Pareto (i.e., $\Gamma_0(z) = 1 - z^{-\frac{1}{\sigma}}$). The parameter σ governs the thickness of the right tail of the productivity distribution. Since the variance of the logarithm of productivity z is precisely σ, I refer to σ as "productivity dispersion." Assumption 1 requires that $\sigma < \frac{1}{1-\alpha}$.

[8]In a slight abuse of language, I refer to the object $\omega(z) \equiv \frac{\chi-\overline{g}}{\chi-g(z)} \frac{y(z)}{Y}$ as the output share of firms with productivity z. To be precise, $\omega(z)$ is a density that satisfies $\int \omega(z)\,\mathrm{d}\Gamma(z) = 1$. Similarly, I refer to $\frac{\chi-\overline{g}}{\chi-g(z)}$ as the employment share of firms with productivity z.

PROPOSITION 6—Pareto special case with no discounting: *Suppose that* $\Gamma_0(z) = 1 - z^{-\frac{1}{\sigma}}$ *for* $z \geq 1$ *and* $r = 0$. *The aggregate labor share is given by*

$$\mathrm{LS} = 1 - \alpha - \underbrace{\frac{\chi_x - \overline{g}}{\chi - \overline{g}}\sigma}_{\text{profit share}}, \tag{30}$$

where $\frac{\chi_x - \overline{g}}{\chi - \overline{g}} \in (0, 1)$, *which implies that* $\mathrm{LS} \in (0, 1 - \alpha)$.

The key takeaway is that, all else constant, the aggregate labor share is decreasing in productivity dispersion σ. The Cobb–Douglas exponent α determines the share of payments to capital while productivity dispersion σ is a key determinant of how the remaining $1 - \alpha$ share of aggregate value-added is split between wages and profits. Notice that free-entry exhausts ex ante rents, so that without productivity dispersion (i.e., $\sigma = 0$), there would be no ex post profits.[9]

To understand why the aggregate labor share is decreasing in productivity dispersion, recall from (26) that the pass-through of productivity to wages is proportional to the amount of local competition that a firm faces (i.e., the density of workers at firms with a similar productivity level). An increase in productivity dispersion σ makes firms become "further apart" in terms of productivity, effectively weakening between-firm competition in the labor market.

Finally, notice that the persistence of the productivity process χ affects both the level of the aggregate labor share as well as its sensitivity to productivity dispersion σ. All else equal, if productivity differentials between firms were purely transitory (i.e., $\chi \to +\infty$), the labor share would converge to the frictionless case $\mathrm{LS} = 1 - \alpha$, where aggregate profits are zero. In that case, productivity dispersion would not affect the labor share. If instead productivity differentials across firms were permanent (i.e., $\chi = 0$), then the labor share would converge to $1 - \alpha - \sigma$. Therefore, what matters for the share of profits in GDP is not only how far apart firms are in terms of productivity, but how persistent these productivity differentials are.

Comparison With Benchmark Models

I now contrast the prediction of the baseline model with two benchmark models. First, I consider an extension of the model in (Burdett and Mortensen (1998)) with capital (henceforth "Burdett–Mortensen"). The main difference with the baseline model is that productivity differentials are permanent instead of being mean-reverting. Second, I consider a variation of the Coles and Mortensen (2016) model with capital and balanced matching (henceforth "Coles–Mortensen"). The only difference with the baseline model is that entry and exit is exogenous. Note that the original model developed in Coles and Mortensen (2016) does not have balanced matching, but instead has a convex hiring cost function. I study the implications of costly hiring (i.e., endogenous vacancy-posting) separately in Section 5.3.

Table I contains closed-form expressions for the aggregate labor share in the Pareto special case with no discounting for each benchmark model. First, notice that the labor

[9]To be precise, the marginal entrant makes zero profits in expectation (i.e., $v(\underline{z}) = 0$ as per Assumption 3). Therefore, if all firms had the same productivity level (i.e., $\sigma = 0$), then the present value of aggregate profits would be zero. In a stationary equilibrium with no discounting, this implies zero aggregate profits.

TABLE I

PARETO SPECIAL CASE WITH NO DISCOUNTING: COMPARISON WITH BENCHMARK MODELS.

Model	Aggregate labor share
Frictionless	$1-\alpha$
Baseline model	$1-\alpha-\frac{\chi_x-\overline{g}}{\chi-\overline{g}}\sigma$
Coles-Mortensen	$1-\alpha-(\frac{\chi_x-\overline{g}}{\chi-\overline{g}}+\frac{\chi_x-\overline{g}}{\chi_x-\underline{g}}\frac{\chi_s}{\chi-\overline{g}})\sigma$
Burdett-Mortensen	$1-\alpha-\sigma$

Note: The models are ranked according to their aggregate labor share, from highest to lowest. See Appendix B for a formal derivation of the Coles–Mortensen and Burdett–Mortensen models. "Productivity dispersion" σ is the inverse of the Pareto exponent; $\overline{g}\equiv\lambda u-\delta$ is the average growth rate of continuing firms; $\underline{g}\equiv-\lambda(1-2u)-\delta$ is the growth rate of firms with $z=\underline{z}$.

share in the baseline model is higher than in the Coles–Mortensen model. The reason is that, in the baseline model, endogenous entry exhausts rents up to the point where the marginal entrant has a continuation value of zero (i.e., $v(\underline{z})=0$). In Coles–Mortensen, entry is exogenous and all firms make weakly positive flow profits, which implies that $v(\underline{z})>0$. As a result, profits are higher in Coles–Mortensen and the labor share is lower.

Second, the labor share in Burdett–Mortensen is the lowest of all models considered. To understand why, notice that the labor share in the Burdett–Mortensen model coincides with the labor share in the baseline model when there is no mean-reversion in productivity (i.e., $\chi=0$). As discussed earlier, a high persistence of productivity differentials reduces the labor share in favor of profits.

The main takeaway from Table I is that both endogenous entry and mean-reversion in firm productivity reduce the amount of rents that accrue to firms. As a result, these model ingredients generate a higher labor share (i.e., closer to the frictionless benchmark) and a lower sensitivity of the labor share to productivity dispersion. Figure 1b contains a numerical example where the aggregate labor share is shown as a function of productivity dispersion (keeping everything else constant).

While the results in Table I are obtained under strong assumptions (i.e., a Pareto distribution and zero discounting), they provide theoretical guidance as to which model parameters are important: the capital share α, the productivity process (σ,χ), and the pace of job reallocation between firms $(\overline{g},\chi_x)$. In the next section, I conduct a quantitative analysis of the model where all functional forms and parameter values are disciplined by moments in the data.

3. QUANTIFYING THE THEORY

3.1. *Calibration*

I calibrate the model to the Canadian economy over the 2000–2015 period. As in the U.S., the Canadian labor share is now lower than it was in the 1980s but, unlike in the U.S., the Canadian labor share appears to have been quite stable since 2000 (see Appendix D.2). I will thus compare the 2000–2015 period to the stationary equilibrium of the model.

Data

I use microdata from the *National Accounts Longitudinal Microdata File* (henceforth "NALMF"), which is constructed by merging administrative data from different sources.

The NALMF contains de-identified data covering the universe of private sector employers in Canada over the 2000–2015 period. The unit of observation is an enterprise, which is an entity larger than an establishment but smaller than a consolidated corporation.[10] Unlike with establishments, these entities have a full set of income statement and balance sheets, which makes it possible to construct firm-level measures of value-added.

I restrict the main sample to the private corporate sector excluding: agriculture, mining, utilities, education, and health (i.e., NAICS 11, 21, 22, 61, and 62). Agriculture and mining are excluded due to data limitations while utilities, education, and health are excluded due to the fact that these sectors are dominated by public entities in Canada. Finally, I restrict the sample to firm-year observations with more than 5 employees that have no missing values in employment, value-added, capital stock, and industry code.

The data allows me to compute value-added at the firm level using the same methodology as in the *System of National Accounts*. Using the income approach, I construct value-added as the sum of worker compensation (i.e., income that accrues to workers) and gross profit (i.e., income that accrues to firm owners). I can then compute the labor share as the ratio of worker compensation to value-added. Labor productivity is defined as the ratio of value-added to employment. In Appendix D.1, I describe the variable construction in more details and provide summary statistics. The final sample contains 3,084,182 firm-year observations, with an average of 192,761 firms per year.

Productivity Distribution

I model the productivity distribution $\Gamma_0(z)$ as a Gamma distribution with a mean normalized to one. The Gamma distribution has both a thick upper tail and a cusp, which is consistent with the typical shape of productivity distributions in the data. The resulting distribution is fully characterized by a single shape parameter $\eta > 0$ that governs the thickness of the upper tail, where a lower η implies a thicker tail.[11] As I show in Appendix C.3, the Gamma distribution provides a good fit for the empirical distribution of productivity, while the Pareto distribution assumed in Proposition 6 generates too much skewness. In Section 5.4, I consider an alternative calibration of the baseline model where $\Gamma_0(z)$ is a Pareto distribution.

Externally Calibrated Parameters

The model has nine parameters. I externally calibrate the first three (r, d, α) and internally calibrate the last six $(\lambda, \mu, \delta, b, \chi, \eta)$. I interpret a time period $[t, t+1)$ in the model to be a year. The value of r is chosen to imply an annual discount rate of 5%. For the depreciation rate d, I obtain a target by using industry-level data from Statistics Canada (Table 031-0006, linear depreciation). For the set of industries covered in the main sample, the average annual depreciation rate over the 2000–2015 period is 14.8%. Finally, I set $\alpha = 0.230$ to match the capital-output ratio of 1.10 measured in the NALMF data. Table II

[10]The full definition taken from *Statistics Canada*'s website is "Enterprise refers to the highest level of the Business Register statistical hierarchy and is associated with a complete set of financial statements. The enterprise, as a statistical unit, is defined as the organizational unit of a business that directs and controls the allocation of resources relating to its domestic operations, and for which consolidated financial and balance sheet accounts are maintained from which international transactions, an international investment position and a consolidated financial position for the unit can be derived. It corresponds to the institutional unit as defined for the System of National Accounts."

[11]The PDF is defined over $\mathbb{R}_+$ and is given by $\Gamma_0'(z) = \frac{\eta^\eta}{G(\eta)} z^{\eta-1} e^{-z\eta}$, where $G(\eta)$ is the Gamma function.

TABLE II

EXTERNALLY CALIBRATED PARAMETERS.

Parameter	Symbol	Value	Target
Interest rate	r	0.049	Annual discount rate of 5%
Depreciation rate	d	0.160	Annual depreciation rate of 14.8%
Capital share	α	0.230	Capital stock to annual output ratio of 1.10

summarizes these choices. To map discrete-time growth rates g_d in the data to continuous-time growth rates g_c in the model, I use the conversion formula $g_c = \log(1 + g_d)$.[12]

Internally Calibrated Parameters

The remaining parameters $\theta \equiv (\lambda, \mu, \delta, b, \chi, \eta)'$ are jointly calibrated. I target moments from the data and choose the set of parameters that minimizes the distance between the model-implied moments and the data. (In Appendix C.3, I derive analytical formulas for the model-implied moments.) To ensure that Assumption 2 is satisfied, I restrict the parameters space to $\Theta \equiv \mathbb{R}^6_+ \cap \{\theta | 0 < \lambda - \delta < \min\{\chi, \mu\}\}$. I target 7 moments related to the dynamics of firm-level productivity and the pace of job reallocation between firms. The moments are: the job creation rate by continuers, the job creation rate by entrants, the job destruction rate by continuers, the job destruction rate by exiters, the unemployment rate, the autocorrelation of firm-level log labor productivity, and the interdecile range of log labor productivity.

I take the unemployment rate, job creation rate by entrants, and job destruction rate by exiters directly from publicly available data (Statistics Canada Tables 282-0087 and 527-0001) and compute 2000–2015 averages. The job creation rate by entrant is defined as the sum of all jobs created by entering firms in a year divided by the total number jobs in the economy at the beginning of the year. The job destruction rate by exiters is defined as the number of jobs destroyed by exiting firms divided by the number of total jobs in the economy.

The reason why I do not use the NALMF microdata to estimate the job creation rate (destruction rate) by entrants (exiters) is because my main sample is not exactly the universe of firms due to sample restrictions. Therefore, firms moving in and out of the sample does not correspond to firm entry and exit. Moreover, it is not possible to accurately distinguish entry and exits from mergers and acquisitions using only the NALMF data.

The remaining moments are computed using the NALMF microdata. The job creation rate by continuers is defined as the sum of jobs created by expanding firms divided by the number of total jobs in the economy. Similarly, the job destruction rate by continuers is defined as the sum of jobs destroyed by shrinking firms divided by the number of total jobs in the economy. The autocorrelation of firm-level log labor productivity is obtained by estimating the autoregressive parameter in a firm-level panel AR(1) regression with year and 3-digit NAICS fixed effects. To compute the interdecile range of log labor productivity, I first remove year and (3-digit NAICS) industry fixed effects from firm-level log labor productivity. I then compute the difference between the 90th and 10th percentiles of the (residualized) log labor productivity in the pooled 2000–2015 sample.

[12] Suppose that a continuous-time process satisfies $\dot{x}_t = g_c x_t$. Solving forward, we have that $\frac{x_{t+1}}{x_t} = e^{g_c}$. The definition of discrete-time growth rate is $g_d \equiv x_{t+1}/x_t - 1$. Combining both equations and solving for g_c, I obtain $g_c = \log(1 + g_d)$.

TABLE III

TARGETED MOMENTS.

Moment	Data	Model
Unemployment rate	0.071	0.071
Autocorrelation of log labor prod.	0.810	0.806
Interdecile range of log labor prod	1.547	1.547
Job creation (continuers)	0.061	0.055
Job creation (entrants)	0.019	0.022
Job destruction (continuers)	0.067	0.062
Job destruction (exiters)	0.016	0.016

Numerical Solution

In Appendix C.1, I provide a detailed description of the solution method. Denote the vector of moments in the data by $\hat{\Lambda}$. For each value of $\theta \in \Theta$, I solve the model numerically and compute the model-implied moments $\Lambda(\theta)$. I then choose the parameter θ that minimizes the following quadratic form:

$$\min_{\theta \in \bar{\Theta}} \left(\Lambda(\theta) - \hat{\Lambda}\right)' \left(\Lambda(\theta) - \hat{\Lambda}\right). \tag{31}$$

Since Θ is not compact, I implement the numerical optimization problem by using its closure $\bar{\Theta}$ and then verify that the minimizer θ^* is in Θ. I obtain the parameter estimate $(\lambda, \mu, \delta, b, \chi, \eta) = (0.2443, 0.3063, 0.0233, 0.5981, 0.2311, 3.8773)$. Table III contains the fit of the model and in Appendix C.3, I report the sensitivity of the targeted moments to changes in parameter values (see paragraph titled "Jacobian matrix").

The only structural parameters that can be interpreted directly are the flow value of unemployment b and the difference $\lambda - \delta$. The average wage in the economy is 1.0606, so b is equal to 56% of the average wage, which is close to (but higher than) the value of 0.4 typically used in the labor search literature. Recall from Lemma 1 that the annual growth rate of employment of a firm is bounded from above by $e^{\lambda-\delta} - 1 \approx 0.25$. The calibration therefore implies that firms cannot grow faster than 25% annually. Strictly speaking, this is counterfactual given that the empirical firm growth distribution typically contains very high values (Bottazzi and Secchi (2006)). Nevertheless, the constraint imposed by search frictions in the model is not overly restrictive. For example, a firm that remains at the top of the productivity distribution for 25 years will grow its employment by a factor of $e^{25\times(\lambda-\delta)} \approx 250$.

3.2. *Validation*

I now validate the model by assessing its ability to match the data along several dimensions. Given that the calibration strategy does not use data on wages or the labor share, everything from now on is nontargeted.

Aggregate and Average Labor Share

As a starting point, consider the empirical counterpart of equation (29),

$$\mathrm{LS} = \sum_i \omega_i \mathrm{LS}_i, \tag{32}$$

TABLE IV

AGGREGATE LABOR SHARE DECOMPOSITION.

Model	Aggregate	Average	Covariance
Data	0.653	0.884	−0.231
Baseline model	0.616	0.889	−0.273
Coles–Mortensen	0.559	0.547	+0.011
Burdett–Mortensen	0.533	0.512	+0.021

which states that the aggregate labor share is the sum of firm-level labor shares LS_i weighted by their value-added share ω_i. Rearranging, the aggregate labor share can be expressed as the sum of the average (unweighted) firm-level labor share and the covariance between labor share and value-added share

$$\underbrace{\mathrm{LS}}_{\substack{\text{Aggregate}\\\text{labor share}}} = \underbrace{\overline{\mathrm{LS}}}_{\substack{\text{Average}\\\text{labor share}}} + \underbrace{\operatorname{cov}(\mathrm{LS}_i, \omega_i).}_{\substack{\text{Covariance between}\\\text{labor share and value-added share}}} \tag{33}$$

In Table IV, I report the three components of equation (33) in the data and in the model. In the data, I compute the components separately in each 2-digit NAICS industry and then take a value-added weighted average. I then compute a simple average over the 2000–2015 period.

The model generates an aggregate labor share of 0.62, which is close but slightly lower than in the data (0.65). The average (unweighted) firm-level labor share in the model is 0.89, which is also close to the data (0.88). The resulting covariance between labor share and value-added share is negative, and stands at −0.27 in the model (−0.23 in the data). Overall, the model replicates the main features of the data: a high average labor share and a negative covariance between value-added and labor share in the cross-section of firms.

Table IV also contains the same decomposition for the Coles–Mortensen and Burdett–Mortensen models (see Appendix B for a description of the calibration strategy). Consistent with the theoretical result in the Pareto special case (see Table I), the labor share is lower in the Coles–Mortensen and Burdett–Mortensen economies. Notice that, in both cases, the covariance between value-added share and labor share is close to zero, while it is strongly negative in the data and in the baseline model. As we will see shortly, this is due to the fact that the benchmark models cannot generate labor shares above $1-\alpha$ (i.e., negative flow profits) for low-productivity firms.

Assessing Cross-Sectional Moments

The negative cross-sectional covariance between labor share and value-added share is not only a feature of the Canadian data. It has previously been documented in the case of the Taiwanese manufacturing sector (Edmond, Midrigan, and Xu (2015)), the U.S. manufacturing sector (Kehrig and Vincent (2021)), and most other sectors of the U.S. economy (Autor et al. (2020)). The value-added share of a firm is high either because it is large (it employs a lot of workers) or because it it has a high labor productivity (it produces a lot of value-added per worker). A key prediction of the model is that the negative relationship between value-added and labor share is entirely driven by productivity, not size (see Proposition 2). I now confront this prediction with the data.

TABLE V

ASSESSING CROSS-SECTIONAL MOMENTS.

	(1)		(2)		(3)	
Labor share (log LS)	Data	Model	Data	Model	Data	Model
Labor productivity ($\log y$)	−0.313 (0.002)	−0.620	−0.313 (0.002)	−0.620		
Employment ($\log N$)			0.009 (0.001)	0	−0.001 (0.001)	−0.007
Industry fixed effects	✓		✓		✓	
Year fixed effects	✓		✓		✓	
Sample size	3,084,182		3,084,182		3,084,182	
R^2	0.272		0.273		0.036	

Note: Standard errors in parentheses are clustered at the firm-level.

Table V contains results from a panel regression of firm-level labor shares on different covariates (all variables are in logarithm) with year and 3-digit NAICS industry fixed effects. Specification (1) regresses labor share on labor productivity. I obtain a negative coefficient of −0.31. As a point of comparison, I compute the (population) regression coefficient in the calibrated model and obtain a value of −0.62. One concern with the regression specification is that the presence of measurement error in value-added could lead to a spurious negative relationship between labor share and labor productivity.[13] In Appendix E.1, I address this concern by instrumenting labor productivity with its 1-year lag. I obtain a slightly lower value in absolute terms (−0.25).

Specification (2) regresses firm-level labor shares on both labor productivity and size. The coefficient on labor productivity remains mostly unchanged (−0.31) while the coefficient on employment is close to zero (<0.01). This is roughly in line with the model predictions, where (i) introducing employment as a control does not affect the coefficient on labor productivity and (ii) the coefficient on employment is zero. Specification (3) regresses firm-level labor shares on size by itself. The relationship is negative but very small (−0.001). In the model, the coefficient is larger in absolute value but also very small (−0.007). Through the lens model, the relationship between employment and labor share is spurious in the sense that it only reflects the positive correlation between productivity and employment in a stationary equilibrium.

The empirical relationship between labor productivity and labor share could in principle be explained by the fact that firms differ in their capital share (i.e., the share of payments to capital in value-added). For example, if a firm purchases a robot to replace some of its workers, then labor productivity will increase (because the number of employees decreases) and the share of payments to capital in value-added will increase (because the stock of capital increases). In Appendix E.1, I estimate Specification (1) controlling for the capital-output ratio and find a negligible role for (within-industry) capital intensity heterogeneity in explaining differences in firm labor shares across firms.

Overall, I find robust support for the idea that high-productivity firms have a lower labor share. In the model, productivity and wages have a rank correlation of one (i.e., high-wage firms are necessarily high-productivity firms). Hence, one would expect that

[13]The labor share is defined as the ratio of worker compensation to value-added while labor productivity is defined as value-added per worker. If there is a measurement error in value-added, there will be a spurious negative correlation between value-added and labor share.

the relationship between average wage (i.e., compensation per worker) and labor share be positive. When I estimate this cross-sectional relationship in the data, I instead obtain a weak *positive* relationship. In Appendix E.2, I use an accounting decomposition to understand why the model fails to match this particular moment. In short, the reason is that the model does not generate enough wage dispersion between firms.

Unpacking the Aggregate Labor Share

I now assess whether the model-implied relationship between labor productivity, value-added share, and labor share is quantitatively consistent with the microdata. Denote by LS_d and ω_d the labor share and value-added share within a labor productivity decile d. The aggregate labor share can be expressed as

$$\mathrm{LS} = \sum \omega_d \mathrm{LS}_d. \tag{34}$$

I construct the labor productivity deciles in the data by sorting firm-year observations by labor productivity within industry-year bins. I then pool industries and years together and compute the measure of interest within each decile (see Appendix D.3 for details).

The left panel of Figure 2 contains the labor share by labor productivity deciles in the model and in the data. First, notice that there are large differences in the labor share across firms. In the model, the difference between the labor share at the top decile versus the bottom decile is 1.22 (1.14 in the data). To provide a benchmark, I include a horizontal line at $1-\alpha$, which corresponds to the labor share that would prevail in a frictionless competitive model. Notice that a large fraction of firms have a high labor share. In the model and in the data, the bottom 5 deciles of firms have a labor share above $1-\alpha$. In contrast, the top decile of firms have a low labor share of 0.52 (0.44 in the data). In line with the predictions of the baseline model (see Proposition 4), labor shares above $1-\alpha$ are a pervasive feature of the data. The Burdett–Mortensen and Coles–Mortensen models instead predict that all firms have a labor share strictly below $1-\alpha$ (see Appendix Figure E.1d).

Both in the model and in the data, value-added is extremely concentrated within high-productivity firms (see right panel of Figure 2). In the model, the top decile of firms account for 55% of of value-added (42% in the data), while the bottom 5 deciles accounts

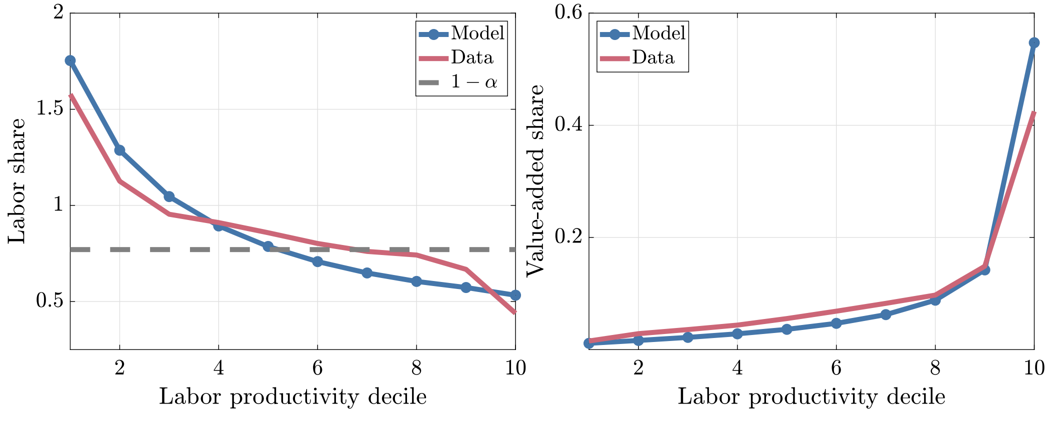


FIGURE 2.—Aggregate labor share decomposition.

for a mere 11% (18% in the data). Through the lens of the model, the negative covariance between labor share and value-added share documented earlier can thus be explained by the fact that labor shares are decreasing in firm productivity while value-added shares are (on average) increasing in firm productivity.

In Appendix E.3, I conduct a number of robustness checks to assess the importance of industries, firm size, capital share heterogeneity, and measurement error in value-added. Overall, I find that the main findings are robust: (i) there is a strong negative relationship between firm-level labor share and labor productivity and (ii) a large fraction of firms have a labor share above $1-\alpha$.

Assessing Panel Moments

I now assess the ability of the model to match panel moments (i.e., the comovement of variables within firms over time). I focus on two core predictions of the model: (i) firms respond to an increase in productivity by increasing their wage (see Proposition 2), which (ii) induces an increase in their employment growth (see Lemma 1).

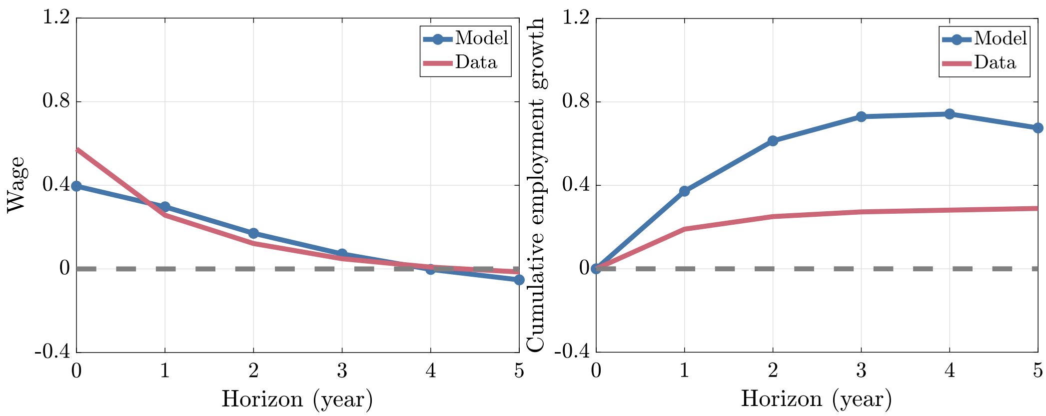
To compare model and data, I use the following auxiliary models:

$$\log w_{i,t+k} = \mu_i^k + \lambda_t^k + \beta_w^k \log y_{i,t} + u_{i,t+k}, \tag{35}$$

$$\Delta^k \log N_{i,t} = \mu_i^k + \lambda_t^k + \beta_g^s \log w_{i,t} + u_{i,t+k}. \tag{36}$$

The parameters μ_i and λ_t denote firm and year fixed effects. The variables $y_{i,t}$, $w_{i,t}$, and $\Delta^k \log N_{i,t}$ denote, respectively, labor productivity, average wage, and k-year ahead log employment growth for firm i in year t. The coefficient β_w^k represents the response of wages at time $t+k$ to an increase of labor productivity at time t. Similarly, β_g^k is the response of employment growth between t and $t+k$ to a wage increases at time t.

Figure 3a plots the dynamic response of wages to an increase in labor productivity over a 5-year horizon, both in the model and in the data.[14] In the data, I do not observe



(a) Labor productivity shock (b) Wage shock

FIGURE 3.—Assessing panel moments (coefficients from specifications (35) and (36)).

[14] Using the calibrated model, I simulate synthetic trajectories of employment, wages, and labor productivity for 200,000 firms. I first simulate 16 years of monthly data, and then aggregate to the annual frequency. I obtain a panel of synthetic data, which I use to estimate (35) and (36) exactly as in the NALMF data.

worker-level wages, so I construct a measure of the average wage of a firm by dividing total worker compensation by employment. On impact, wages increase less than one-for-one with labor productivity ($\beta_w^0 = 0.40$ in the model and $\beta_w^0 = 0.57$ in the data). The interpretation is that, on average, workers capture roughly half of productivity increases through higher wages. After 4 years, wages have fully reverted back to zero both in the model and in the data, reflecting the fact that productivity shocks are mean-reverting.

Figure 3b plots the dynamic response of cumulative employment growth to an increase in wages. Both in the model and in the data, cumulative employment growth increases strongly over the first 2 years and then stabilizes. At the one-year horizon, the interpretation is that when a firm increases its wage by 10%, its annual growth rate of employment increases by 3.7 percentage points in the model and 1.9 in the data. The model thus overpredicts the employment response to a wage increase. Taken at face value, this divergence between model and data means that the model does not generate enough monopsony power. To see that, notice that the first-order condition for wages (14) implies that $1 = v(z)\widetilde{g}'(w)$. Therefore, a high response of employment growth to a wage change $\widetilde{g}'(w)$ must coexist with a low present value of rents accruing to firms $v(z)$ (i.e., a low level of monopsony power).

The relationship between labor productivity and wages that I estimate is related to existing empirical evidence on the wage-productivity pass-through. The empirical literature consistently finds pass-through elasticities that are positive and well below one (see Card, Cardoso, Heining, and Kline (2018) for a literature review). Similarly, my estimates of the response of employment growth to wages is related to the literature studying separation elasticities (see Manning (2011) for a literature review).[15]

It is worth noting that my dynamic estimates correspond to correlations in the data and do not necessarily have a causal interpretation. To address this, I confront the model with existing causal evidence. Kline, Petkova, Williams, and Zidar (2019) is the most relevant study since it estimates both the elasticity of labor productivity to wages (pass-through elasticity) and the elasticity of worker separations to wages (separation elasticity) in the cross-section of U.S. firms. The identification strategy is based on comparing firms whose patent applications were initially allowed to those whose applications were initially rejected. They find that firms respond to rising productivity by increasing worker compensation, which induces a decline in worker separations. In Table VI, I

TABLE VI

MODEL PREDICTIONS AND EVIDENCE FROM KLINE ET AL. (2019).

Statistic	Data	Model
Pass-through elasticity	0.47	0.54
Separation elasticity	−1.62	−1.49

Note: The pass-through elasticity is obtained from Column 2 of Table VIII. It is defined as the percentage change in wage bill per worker in response to a 1% change in value-added per worker. The separation elasticity is obtained from Column 1 of Table IX. It is defined as the percentage change in the fraction of workers who separate during the next year in response to a 1% change in the wage. In Appendix C.3, I provide expressions for the analogous model-implied moments.

[15]The existing literature mostly studies the effect of wages on the separation rate rather than employment growth (which is the difference between the hiring rate and the separation rate). One potential reason is that the hiring rate is often thought to depend not only on the wage, but also on recruiting intensity (usually modeled as vacancy-posting in search model). In Section 5, I study an extension of the model with endogenous vacancy-posting.

replicate the estimates from Kline et al. (2019) in the model. The model matches both moments closely: the pass-through elasticity in the model is slightly higher (0.54 versus 0.47) and the separation elasticity is slighly lower in absolute value (-1.49 versus -1.62).

Wage Distribution

One of the key predictions of wage-posting models is that wage dispersion can be sustained in equilibrium (i.e., the law of one price does not hold). As a final validation exercise, I compare the wage distribution in the model and in the data. I use data from Bowlus, Gouin-Bonenfant, Liu, Lochner, and Park (2021) on the distribution of annual earnings in Canada over the 2000–2015 period and compare it with the distribution of wages *across workers* in the calibrated model.

The variance of log wages is 0.73 in the data, while it is only 0.15 in the model (see Table VII). Most noticeably, the model can not explain the amount of right-tail inequality in the data: the P99-P90 percentile difference is 1.32 in the data, while it is only 0.34 in the model. Since the model does not include worker heterogeneity, this is not surprising. Following the work of Abowd, Kramarz, and Margolis (1999)—who use two-way fixed effect regressions (i.e., AKM regression) to decompose earnings inequality across workers—researchers have found that "worker fixed effects" are the key determinant of earnings inequality.

Both in the US and in Canada, recent studies employing an AKM framework have found that firms account for roughly 10% of the variance of log earnings (see Song, Price, Guvenen, Bloom, and Von Wachter (2019) for the U.S. and Dostie, Li, Card, and Parent (2020) for Canada). According to this measure, the model fits the data much more closely: the variance of log wages attributable to firms is 0.15 in the model versus 0.07 in the data.[16] Finally, Table VII reports the firm-size wage premium in the model and in the data. As in the data, the model predicts a small premium: firms with 10% more employees pay wages that are on average 0.1% higher (0.3% higher in the data).

TABLE VII

WAGE DISTRIBUTION.

Statistic	Data	Model
Log wage distribution		
Total variance	0.73	0.15
P99-P90	1.32	0.34
P90-P50	0.78	0.71
P50-P10	0.75	0.29
Variance of firm fixed effects	0.07	0.15
Firm-size wage premium	0.03	0.01

Note: Data on the wage distribution is obtained from Bowlus et al. (2021) (annual earnings, 2000–2015 average); Firm-size wage premium is obtain by regression firm-level average wage on employment (both in logarithms, $N = 3{,}084{,}182$).

[16]In Section 4, I consider an extension of the model with endogenous recruiting intensity that improves the fit of the model. In particular, the model-implied variance of log wages goes down from 0.15 to 0.12.

TABLE VIII

MOTIVATING EVIDENCE.

	Labor share		Productivity dispersion	
Year	Canada	United States	Canada	United States
2001–2005	0.646	0.675	1.82	2.04
2006–2010	0.646	0.650	1.83	2.14
2011–2015	0.650	0.616	1.84	2.27
Change (%)	0.4	–5.8	0.01	0.23

Note: The labor share is calculated using BEA Table 1.14 (corporate sector) in the U.S. and Statistics Canada Table 3800063 in Canada. In both cases, I compute the aggregate labor share as the ratio of "compensation of employees" to the sum of "compensation of employees" and "gross operating surplus." Productivity dispersion is calculated using Compustat data for the U.S. and the NALMF for Canada. In both cases, I compute the interdecile range of (log) sales per worker net of 2-digit NAICS and year fixed effects, excluding industries 11, 21, 61, and 62.

4. PRODUCTIVITY DISPERSION AND THE LABOR SHARE

I now use the calibrated model to simulate the effect of a rise in productivity dispersion.

Motivating Evidence. To motivate the exercise, I first present data on productivity dispersion and the labor share in Canada and the U.S. over the the 2000–2015 period (see Table VIII). For comparability, I use publicly available data on the labor share from Statistics Canada and the BEA. In both countries, I compute productivity dispersion as in Section 3.1, but using sales per worker as a proxy for labor productivity.[17] I use Compustat data for the U.S. and the NALMF for Canada. In Appendix D.4, I validate my estimate of productivity dispersion for the U.S. against Census microdata from Barth et al. (2016).

One striking observation is that over the 2000–2015 period, the labor share has declined by nearly 6 percentage point in the U.S. while it has remained essentially unchanged in Canada. The mirror image is true for productivity dispersion: the interdecile range of sales per worker has increased by 23 log points in the U.S. but by only 1 log point in Canada.

4.1. *Model Experiment*

I now use the model to simulate the effect of a rise in productivity dispersion of the same magnitude as what the U.S. economy as experienced over the 2000–2015 period. To do so, I lower the parameter η from 3.8773 to 2.9974, which implies a rise in the interdecile range of labor productivity increases by 23 log points. All other parameters remain unchanged.

The model experiment consists of comparing the stationary equilibrium of the model before and after the shock. Table IX summarizes the effect of this "productivity dispersion shock." First, the aggregate labor share declines by 1.3 percentage points. To put into perspective, a permanent 1.3% decline in the labor share is quite large, representing roughly a quarter of the 5.8% decline of the U.S. labor share reported in Table VIII. In my framework, the capital-output ratio would need to rise by 6.3% in order to generate such a decline of the labor share.[18] Interestingly, the average (unweighted) labor share moves in

[17]The reason why I use sales per worker instead of labor productivity is that firm-level value-added data does not exist in the U.S. outside of the manufacturing sector.

[18]In my calibration, the user cost of capital is $R = 20.9\%$ and the capital share $R \times \frac{K}{Y}$. If $\frac{K}{Y}$ increases by 6.4%, the labor share would decline by $0.209 \times 6.4\% \approx 1.3\%$.

TABLE IX

RESPONSE OF AGGREGATES (MODEL EXPERIMENT).

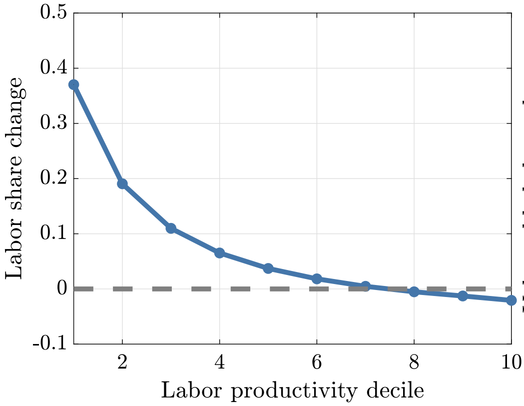
Variable	Steady-state	Counterfactual	Change (pp.)
Labor share	0.616	0.603	−1.33
Average labor share	0.889	0.964	7.50
Covariance	−0.273	−0.361	−8.83
Capital share	0.230	0.230	0.00
Profit share	0.154	0.167	1.33
Output	1.598	1.714	7.26
Unemployment	0.071	0.071	−0.01

the opposite direction, increasing by 7.5%. The capital share remains unchanged, so the decline of the labor share is exactly compensated by a rise of the profit share. The rising wedge between average and aggregate labor share implies that the covariance between labor share and value-added share increases (in absolute value).

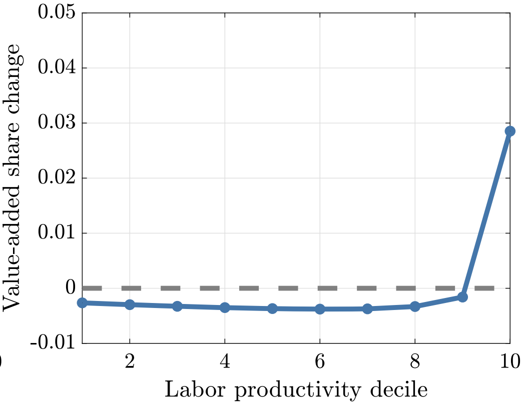
Recall that the productivity distribution $\Gamma_0(z)$ has a mean normalized to one, so a change in its shape parameter η represents a mean-preserving spread. However, given that employment is concentrated within high-productivity firms, the mean-preserving spread generates an increase in aggregate output of 7.3%. The unemployment rate remains essentially unchanged, reflecting the fact that the rise in productivity dispersion has little effect on firm entry and exit. (This is consistent with the Jacobian matrix in Appendix Table C.II, which indicates that the only targeted moment that is meaningfully responsive to a change in η is productivity dispersion.)

Adjustment and Reallocation Effects

To highlight the forces at play, I now study the impact of the productivity dispersion shock along the productivity distribution. Figure 4 plots the response of labor shares and value-added shares by labor productivity deciles. The positive response of the average (unweighted) labor share reported in Table IX can be explained by the fact that,



(a) Labor shares



(b) Value-added shares

FIGURE 4.—Response of labor shares and value-added shares (model experiment).

for all but the top 3 deciles, firm labor shares *increase*. In contrast, the labor share of high-productivity firms decrease.

The intuition is that firms in the right tail of the productivity distribution become further apart in terms of productivity, which effectively shields them from wage competition. As a result, they keep a larger share of value-added as profits. On the other end of the spectrum, low productivity firms are now willing to absorb larger losses in order to retain their workers. The reason is that the rising profit margins of high-productivity firms lead to an increase of the option value of retaining workers.

While employment shares remain unchanged (see Proposition 3), the relative productivity of firms at the top of the distribution increases, so their value-added share increases. Figure 4b indicates that the bottom 9 deciles of firms lose value-added shares at the expense of the top decile. Taking stock, the model predicts that most firms see an upward *adjustment* of their labor share, but that there is a *reallocation* of value added toward high-productivity firms who have a low labor share.

To quantify the relative importance of the adjustment and reallocation components, I use a shift-share decomposition as in Baily, Hulten, and Campbell (1992). Using equation (34), the change in the aggregate labor share ΔLS can be expressed as the sum of three terms:

$$\underbrace{\Delta \mathrm{LS}}_{\text{Total }(-1.33\%)} = \underbrace{\sum \omega_d \Delta \mathrm{LS}_d}_{\text{Adjustment }(+0.03\%)} + \underbrace{\sum \Delta\omega_d \mathrm{LS}_d}_{\text{Reallocation }(-1.08\%)} + \underbrace{\sum \Delta\omega_d \Delta \mathrm{LS}_d}_{\text{Directed reallocation }(-0.29\%)}, \tag{37}$$

where ω_d and LS_d denote the value-added share and labor share of firms in productivity decile d. The adjustment term measures the effect of changing the labor shares LS_d while keeping the value-added shares ω_d constant. It accounts for a small positive increase in the aggregate labor share (+0.03%). The reallocation term measures the effect of changing the value-added shares ω_d while keeping the labor shares LS_d constant. The reallocation component is the dominant force pushing the labor share down (−1.08%). Finally, Kehrig and Vincent (2021) suggest a "directed reallocation" interpretation of the third term. It accounts for the comovement of labor shares and value-added shares. Since there is reallocation of value-added toward firms with a *declining* labor share, the directed reallocation term is negative (−0.29%).

Related Empirical Evidence

The model provides a rich set of predictions regarding the effect of a productivity dispersion shock and many of them can be related to existing empirical studies. First, the decline of the labor share in the model is driven by a reallocation of value-added toward firms with a low (and declining) labor share, and is partially offset by an *increase* in the labor share of most firms. This is precisely what Kehrig and Vincent (2021) document in the U.S. manufacturing sector. They show that over the 1967–2012 period, the aggregate labor share declined by 4.5 percentage point per decade, while the median labor share increased by 0.7 percentage point per decade. They reconcile this divergence by showing that low-labor-share firms have gained value-added shares over time. Over this same period, Kehrig (2015) documents a secular increase in TFP dispersion in the U.S. manufacturing sector. Similarly, Autor et al. (2020) study all U.S. industries and find that falling labor shares are largely accounted for by reallocation rather than a fall in the (unweighted) average labor share across firms.

Second, the model predicts that, in response to a productivity dispersion shock, the concentration of value-added increases while employment concentration remains constant. Consistent with this prediction, Kehrig and Vincent (2021) find that there has been a massive reallocation of value-added shares toward low-labor-share firms, but that there was limited reallocation of employment. At the aggregate level, there has been a secular increase in sales concentration in the U.S. over the past two decades (Grullon, Larkin, and Michaely (2019)) that has coincided with the decline of the labor share. The negative comovement between labor share and concentration at the industry-level is often interpreted as evidence in favor of rising monopoly power (Barkai (2020), Autor et al. (2020)). In my model, however, a rise in productivity dispersion generates a negative comovement of these two variables despite the fact that concentration has no causal effect on the labor share.

4.2. *Supportive Evidence*

The existing literature highlights the fact that there has been substantial heterogeneity in labor share dynamics across countries (see Gutiérrez and Piton (2020)) and industries (see Autor et al. (2020)). How much of this can be explained by differential trends in productivity dispersion? To answer this question, I now estimate the relationship between labor share and productivity dispersion using cross-industry and cross-country evidence. Specifically, I estimate the following specifications:

$$\mathrm{LS}_{j,t} = \mu_t + \beta\sigma_{j,t} + \varepsilon_{j,t}, \tag{38}$$

$$\mathrm{LS}_{j,t} = \mu_t + \gamma_j + \beta\sigma_{j,t} + \varepsilon_{j,t}, \tag{39}$$

$$\Delta^5\mathrm{LS}_{j,t} = \mu_t + \gamma_j + \beta\Delta^5\sigma_{j,t} + \varepsilon_{j,t+5}, \tag{40}$$

where σ denotes productivity dispersion (i.e., the interdecile range of log labor productivity), LS denotes the labor share, μ_t is a year fixed effect, γ_j is a unit fixed effect (industry or country depending on the data set used), and Δ^5 denotes a 5-year difference.

The first specification (equation (38)) uses cross-sectional variation of levels for identification. The second specification (equation (39)) uses variation in LS and σ within an industry/country over time for identification. For the last specification (equation (40)), I use non-overlapping 5-year periods to calculate changes, so the identification comes from the low frequency comovement of LS and σ. Barkai (2020) and Autor et al. (2020) both use a version of specification (40) to estimate the effect of concentration on the labor share.

Cross-Industry Evidence

The relative stability of productivity dispersion at the aggregate level in Canada hides a lot of heterogeneity at the industry level. I now leverage this cross-industry heterogeneity to estimate the relationship between labor share and productivity dispersion. To do so, I aggregate the NALMF microdata to construct a panel data set covering most 3-digit NAICS industries over the 2000–2015 period (see Appendix D.5 for summary statistics).

If the Canadian labor market was perfectly segmented along industry lines (i.e., firms within an industry only compete with each other for workers), then each industry could be treated as a separate labor market, and the model predictions would be directly comparable to the cross-industry evidence. In practice, however, workers often move across industries over their career (see Jarosch, Nimczik, and Sorkin (2019) for empirical evidence from Austria). Nevertheless, one would expect the labor share in an industry to be negatively related to its productivity dispersion as long as the labor market is at least

TABLE X

CROSS-INDUSTRY REGRESSIONS.

LS	(1)	(2)	(3)	(4)	(5)	(6)
σ	−0.112	−0.118	−0.063	−0.094	−0.096	−0.071
	(0.029)	(0.047)	(0.034)	(0.026)	(0.023)	(0.020)
HHI				−0.137	−0.533	−0.041
				(0.218)	(0.071)	(0.095)
K/Y				−0.029	−0.022	−0.026
				(0.218)	(0.071)	(0.095)
Year FE	✓	✓		✓	✓	
Industry FE		✓	✓		✓	✓
N	1104	1104	207	1104	1104	207
R^2	0.154	0.736	0.228	0.188	0.776	0.331

Note: Standard errors in (1), (2), (4), and (5) are clustered at the industry level. "LS" denotes labor share; "σ" denotes productivity dispersion (i.e., the interdecile range of log labor productivity); "K/Y" denotes the capital to output ratio. For specifications (3) and (6), the windows are 2000–2005, 2005–2010, and 2010–2015.

partially segmented along industry lines. (In Appendix B.4, I make this point formally in a simplified version of the baseline model.)

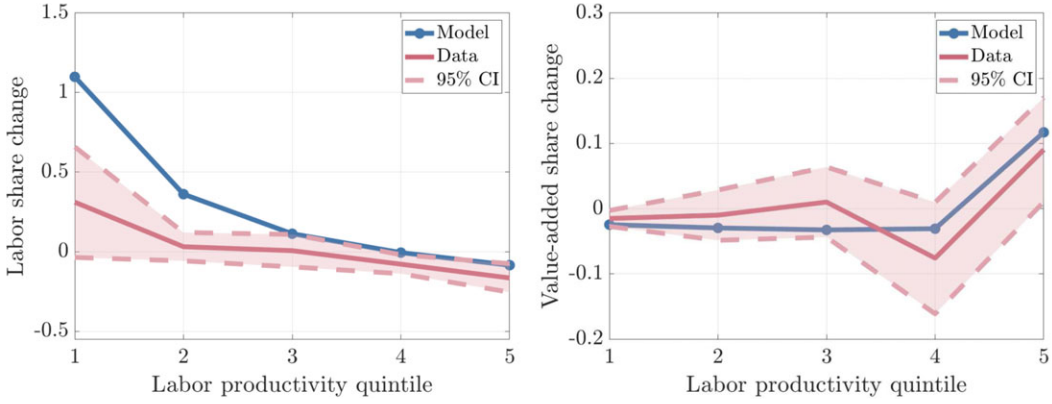
In line with the model experiment, I find that industries with high (rising) productivity dispersion systematically have a low (declining) labor share. Panels (1), (2), and (3) of Table X regress industry-level labor shares on productivity dispersion using specifications (38), (39), and (40). In all specification, I obtain negative coefficients ranging from −0.063 (specification (39)) to −0.118 (specification (39)). As a frame of reference, the implied slope in the model experiment is $-0.0133/0.23 = -0.058$. Figures 6a and 6b contain scatter plots summarizing the relationship between labor share and productivity dispersion (in levels and in 10-year changes).

Other factors such as capital deepening (Karabarbounis and Neiman (2014)) and concentration (Barkai (2020); Autor et al. (2020)) have been show to affect the labor share. To control for these other factors, I reproduce specifications (38), (39), and (40) by adding the capital-output ratio and the HHI (sum of squared firm-level sales shares) as controls (see panels (4), (5), and (6) of Table X). Consistent with existing findings, the coefficients on the HHI and the capital-output ratio are all negative. However, the addition of controls does not materially affect my main findings. If anything, the effect of productivity dispersion becomes more precisely estimated—that is, lower standard errors and less variation across specifications—and the estimated slopes are closer to the model-implied value of −0.058.

I now provide a more stringent evaluation of the model's mechanism. In the model experiment, a rise in productivity dispersion leads to a decline of the aggregate labor share that operates through a reallocation of value-added toward high-productivity firms, as opposed to a broad-based decline of firm-level labor shares. In each industry and year, I sort firms into labor productivity quintiles. For each productivity quintile q, I estimate the following regression:

$$Y_{q,j,t} = \mu_t + \gamma_j + \beta_q \sigma_{j,t} + \varepsilon_{q,j,t}, \tag{41}$$

where μ_t is a year fixed effect and γ_j is a 3-digit NAICS industry fixed effect. $Y_{q,j,t}$ represents the variable of interest (labor share or value-added share) in productivity quintile q, industry j, and year t.



(a) Labor shares (b) Value-added shares

FIGURE 5.—Response of labor shares and value-added shares (model experiment versus data).

Figure 5 plots the estimated coefficients with their 95% confidence intervals (standard errors clustered at the industry level). In panel (a), we can see that the labor share of low-productivity firms increases with productivity dispersion while the opposite is true for high-productivity firms. The coefficients are imprecisely estimated, but for the two top quintiles they are negative and significant at the 5% level. To compare with the model predictions, Figure 5 contains model-implied slopes, which are computed by dividing the change of the variable of interest (i.e., labor share of value-added share) within a specific productivity quintile by the change in productivity dispersion in the model experiment.

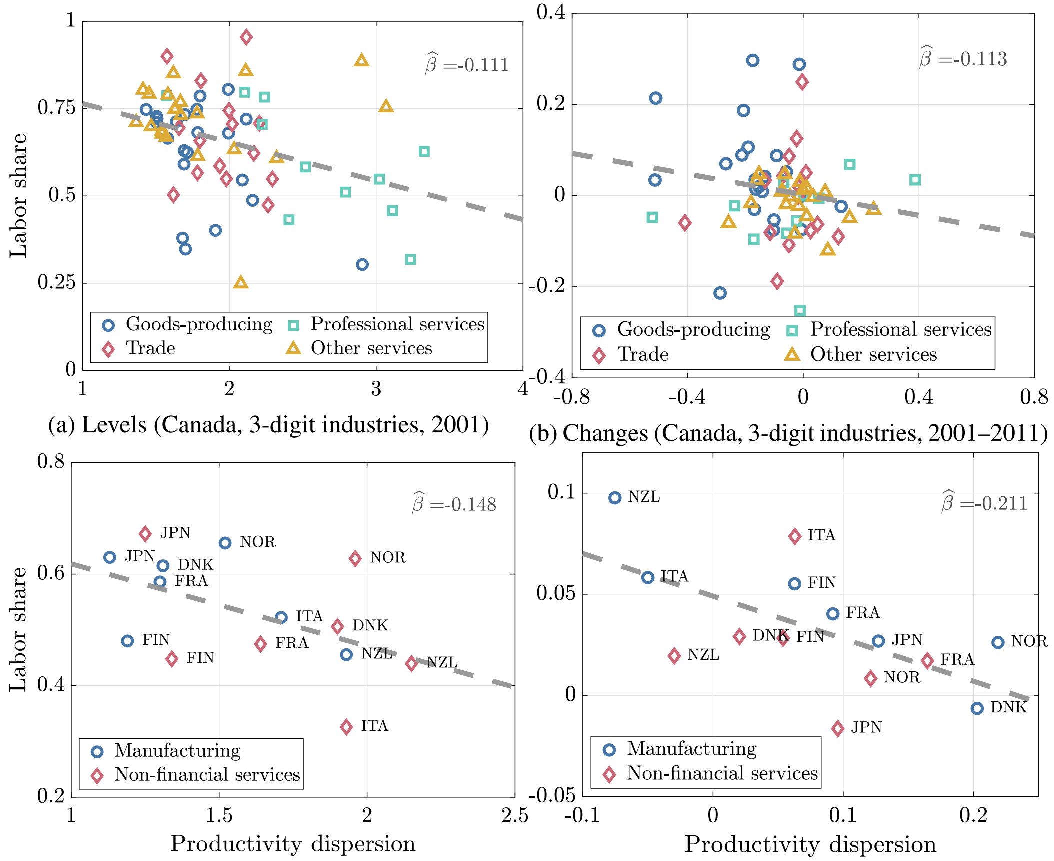
Turning to the reponse of value-added shares (panel (b) of Figure 5), the coefficient for the top quintile of firms is positive while it is negative or near-zero for all the other quintiles. Only the top quintile is significant at the 5% level. Taken as a whole, these findings are remarkably consistent with the transmission mechanism in the calibrated model.

Cross-Country Evidence

Finally, I repeat the exercise using cross-country data. For the cross-country exercise, I use harmonized data on productivity dispersion produced by the OECD.[19] The measure of productivity dispersion in the data set is exactly the same as I used for the cross-industry exercise (i.e., the interdecile range of log labor productivity in the cross-section of firms). The measures are computed at the NACE two-digit industry level and then averaged with employment weights to provide, for each country-year, a measure for the manufacturing sector and for nonfinancial services sectors. I merge the productivity dispersion data with publicly-available data on sectoral labor shares.[20] I obtain a balanced panel of 7 countries (Denmark, Finland, France, Italy, Japan, Norway, and New Zealand) and two broad in-

[19]The OECD has developed a project called *MultiProd* that seeks to provide harmonized cross-country data on productivity and wage dispersion. The data they collect is obtained by running a standardized routine on individual country's production surveys and business registers. The variables that I use (productivity dispersion) is taken directly from the replication material of Berlingieri, Blanchenay, and Criscuolo (2017).

[20]The data on the labor share by sector is obtained from EUKLEMS project (Denmark, Finland, France, and Italy), WORLDKLEMS (Japan), and directly from national statistics institutes (Norway and New Zealand). For a description of the methodology underlying the EUKLEMS and WORLDKLEMS data, see respectively O'Mahony and Timmer (2009) and Jorgenson (2012).



(a) Levels (Canada, 3-digit industries, 2001)

(b) Changes (Canada, 3-digit industries, 2001–2011)

(c) Levels (Cross Country, 2001)

(d) Changes (Cross Country, 2001–2011)

FIGURE 6.—Productivity dispersion and the labor share across countries and industries.

dustry groups (manufacturing and nonfinancial services) for the 2001–2011 period. One year of data was missing for France and Japan, so I used linear interpolation.

Figures 6c and 6d contain scatter plots summarizing the relationship between labor share and productivity dispersion (in levels and in 10-year changes). Consistent with the model predictions, I find that countries with high (rising) productivity dispersion systematically have a low (declining) labor share. In Appendix E.4, I also report the regression results using specifications (38), (39), and (40).

5. EXTENSIONS

I now consider a number of model extensions in order to assess how sensitive the results of the model experiment in the previous section are to different modeling choices. Overall, I find that the baseline model is somewhat conservative in the sense that most of the model extensions considered imply a larger labor share decline in response to a rise in productivity dispersion. I organize the results in Table XI, which contains the steady-state aggregate labor share across model extensions as well as the changes in output, unemployment rate, and labor share induced by a rise in productivity dispersion (i.e., a 23-log point increase in the interdecile range of labor productivity, exactly as in Section 4).

TABLE XI

MODEL EXPERIMENT: EXTENSIONS AND ROBUSTNESS CHECKS.

Model	Steady-state LS	Experiment (% change) Δ Output	Δ Unemployment	Δ LS
Benchmark models				
Baseline model	0.616	+7.26	−0.01	−1.33
Coles–Mortensen	0.559	+7.30	0	−1.57
Burdett–Mortensen	0.533	+10.2	0	−1.97
Model extensions				
Baseline + endogenous b	0.616	+7.30	0	−1.34
Baseline + endogenous λ	0.622	+15.0	+0.22	−1.32
Baseline + endogenous (λ, b)	0.622	+14.7	0	−1.27
Pareto distribution	0.387	+24.7	−2.78	−2.20
Pareto distribution + endogenous b	0.387	+25.2	0	−2.36

5.1. *Benchmark Models*

First, I replicate the model experiment using the calibrated Burdett–Mortensen and Coles–Mortensen models described in Appendix B. Consistent with the theoretical results in the Pareto special case with no discounting (see Table I), the change in the labor share induced by an increase in productivity dispersion in the baseline model (−1.3%) is smaller in absolute value than in the Coles–Mortensen model (−1.6%), which itself is smaller than in the Burdett–Mortensen model (−2.0%).

5.2. *Endogenous-b Extension*

In the baseline model, there is small decline of the unemployment rate in response to a rise in productivity dispersion (see Table IX). The reason is that the equilibrium entry threshold $\underline{z}$ responds to the increased profitability associated with becoming a highly-productive firm. In contrast, the benchmark models do not have that force (i.e., firm entry is exogenous).

For complete comparability with the benchmark models, I now consider an extension of the baseline model where I shut down the equilibrium response of the unemployment rate (henceforth "endogenous-b extension"). To do so, I allow the flow value of unemployment b to depend on all the model parameters θ. In particular, I assume that $b(\theta)$ is such that the equilibrium unemployment rate is always 0.0713 (i.e., the unemployment rate in the steady state of the baseline model).[21]

Repeating the model experiment, I find that the response of the labor share in the endogenous-b extension is essentially the same as in the baseline model (see Table XI). The reason is that the unemployment rate response in the baseline model is negligible (−0.01 percentage point). As we will see shortly, shutting down the response of the unemployment rate will be more important in other model extensions.

[21] To be precise, the flow value of unemployment b is now a function of all the model parameters θ and the equilibrium unemployment rate is a function of exogenous parameters θ as well as b. The function $b(\theta)$ is thus implicitly defined as $u(\theta, b(\theta)) = 0.0713$.

5.3. *Endogenous-λ Extension*

The balanced matching assumption in the baseline model implies that the rate at which firms meet workers is exogenous. As a result, the only way that firms can increase their hiring rate is by increasing their wage. I now consider a model extension where firms can increase the rate at which they meet workers by posting more vacancies (henceforth "endogenous-λ extensions"). In this section, I describe the main equations and discuss the calibration strategy. In Appendix B.3, I provide a complete derivation of the model.

Environment

From now on, I interpret the meeting rate λ in the baseline model as a vacancy rate (i.e., the measure of vacancies per employee). I assume that firms choose λ optimally subject to a convex cost function $c(\lambda, z)$. The employment growth function (equation (18) in the baseline model), is now given by

$$\widetilde{g}(\lambda, w) = \overbrace{\underbrace{\lambda}_{\text{vacancy rate}} \times \underbrace{\widetilde{f}(w)}_{\text{vacancy yield}}}^{\text{hiring rate}} - \overbrace{\big(\Lambda\big(1 - \widetilde{Q}(w)\big) + \delta\big)}^{\text{separation rate}}, \tag{42}$$

where the "vacancy yield" is equal to the probability that a job offer is accepted (i.e., $\widetilde{f}(w) \equiv u + (1-u)\widetilde{P}(w)$). As in the baseline model, $\widetilde{P}(w)$ denotes the wage distribution across workers. The new object Λ denotes the aggregate vacancy rate and $\widetilde{Q}(w)$ denotes the vacancy-weighted wage distribution. The key difference with the baseline model is that firms can increase their hiring rate either through a higher vacancy rate λ, or through a higher wage, which increases the vacancy yield $\widetilde{f}(w)$.

For reasons that will become clear shortly, I consider an isoelastic vacancy-posting cost function of the form

$$c(\lambda, z) = \frac{\underline{\lambda}^{-\theta}}{1+\theta} \frac{v(z)}{1-\alpha} \lambda^{1+\theta}. \tag{43}$$

The parameters $\underline{\lambda}, \theta > 0$ govern, respectively, the level and convexity of the vacancy-posting cost function. The scaling factor $\frac{1}{1-\alpha}$ ensures that the capital share is equal to $1-\alpha$ for all firms, as in the baseline model. Notice that the cost function is proportional to the present value of a hire $v(z)$. The interpretation is that firms with a vacancy rate λ have to pay a flow of real resources equal to a fraction $\frac{\underline{\lambda}^{-\theta}}{1+\theta} \frac{1}{1-\alpha} \lambda^{1+\theta}$ of their value as "vacancy-posting costs."

Equilibrium

As in the baseline model, wages $w(z)$ are increasing in firm productivity and firm entry and exit is determined by an endogenous threshold $\underline{z}$. The following proposition highlights the equilibrium behavior of firms as it relates to hiring.

PROPOSITION 7: *The equilibrium vacancy yield and vacancy rate are given by*

$$f(z) = u + (1-u)P(z), \qquad \text{(Vacancy yield)}$$

$$\lambda(z) = \underline{\lambda} f(z)^{\frac{1}{\theta}}. \qquad \text{(Vacancy rate)}$$

The equilibrium vacancy yield $f(z) \equiv \widetilde{f}(w(z))$ is increasing in firm productivity z, owing to the fact that wages are increasing in productivity. The vacancy rate $\lambda(z)$ is also increasing in productivity. The reason is that a high vacancy yield induces high-productivity firms to post more vacancies per worker. Notice that the parameter $\underline{\lambda}$ governs the level of vacancy-posting while the parameter θ determines the sensitivity of the vacancy rate to firm productivity. The baseline model is nested as a special case where deviation from $\lambda(z) = \underline{\lambda}$ are infinitely costly (i.e., $\theta \to \infty$), which implies that all firms have the same vacancy rate.

Calibration

To calibrate the new parameter θ, I use external evidence on firm vacancies from Davis, Faberman, and Haltiwanger (2013) (henceforth DFH). Using establishment-level data on vacancies in the US over the 2002–2006 period, DFH shows that high-growth firms have both a higher vacancy rate (i.e., number of vacancies per employee) and a higher vacancy yield (i.e., number of hires per vacancy). However, they estimate that the vacancy yield is much more elastic, with respect to the hiring rate, than the vacancy rate. The following proposition clarifies how I map the DFH evidence to the model.

PROPOSITION 8: *The equilibrium cross-sectional elasticities of the vacancy rate λ and vacancy yield f to the hiring rate h are given by*

$$\frac{d \log \lambda}{d \log h} = \frac{1}{1+\theta}, \qquad \frac{d \log f}{d \log h} = \frac{\theta}{1+\theta}.$$

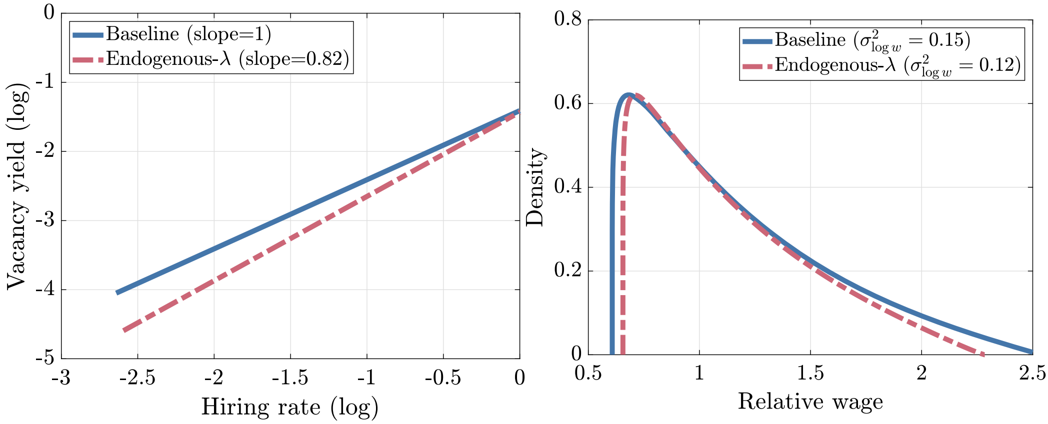
Consider two extreme cases. If the vacancy cost function is linear (i.e., $\theta = 0$), then differences in hiring rates across firms are entirely due to differences in the vacancy rate (i.e., $d \log \lambda / d \log h = 1$). If instead, the cost function is infinitely convex (i.e., $\theta \to +\infty$), then differences in hiring rates across firms are entirely due to differences in the vacancy yield (i.e., $d \log f / d \log h = 1$). In short, θ determines the relative importance of the vacancy rate and the vacancy yield in explaining the heterogeneity in hiring rates across firms.

DFH estimates a robust log-linear relationship between the vacancy yield and the hiring rate with an elasticity of $d \log f / d \log h = 0.82$ (see Figure IX in Davis, Faberman, and Haltiwanger (2013)). The choice of the cost function (43) is thus justified ex post by the fact that it generates an exact log-linear relationship between these two variables. The idea of backward engineering the cost function to match the evidence from DFH has previously been used in Gavazza, Mongey, and Violante (2018).

For the calibration, I first set $\theta = \frac{0.82}{1-0.82} = 4.56$ and then jointly calibrate the model parameters $(\underline{\lambda}, \theta, \mu, \delta, b, \chi, \eta)$ by targeting the same 7 moments as in the baseline model. In Appendix C.1, I extend the solution algorithm to account for endogenous vacancy-posting and in Appendix C.4, I report the targeted moment fit.

Quantitative Results

The aggregate labor share in the endogenous-λ extension is about one percentage point higher than in the baseline model (see Table XI), entirely due to a lower profit share (the capital share is $\alpha = 0.23$ in both models, see Appendix B.3 for a discussion). Moreover, the aggregate labor share responds somewhat more mildly to a productivity dispersion shock than in the baseline model (see Table XI). Shutting down the equilibrium response of the unemployment rate (i.e., row "endogenous-(λ, b) extension" of Table XI), the response is slightly lower in absolute value.



(a) Davis–Faberman–Haltiwanger plot

(b) Wage distribution

FIGURE 7.—Hiring and wages in the endogenous-λ extension (calibrated model).

This finding suggests that the addition of a vacancy-posting margin effectively reduces the amount of monopsony power. A related finding emerges in Manning (2006), who studies a simple model where monopsonistic firms can raise employment either by increasing the wage offered or increasing expenditures on recruiting. The author shows that the convexity of the recruiting cost function is a central determinant of the amount of monopsony profits.

Quantitatively, I find that moving from $\theta \to \infty$ (in the baseline model) to $\theta = 4.56$ (in the endogenous-λ extension), implies only a modest decrease in the profit share of about 1 percentage point. To understand why, it is important to go back to the micro moment that pins down θ. Figure 7a plots the equilibrium hiring rate against the vacancy yield, both in logarithms. While the endogenous-λ extension has a slope lower than in the baseline model (0.82 versus 1), the slope is still well above 0, indicating that high-growth firms use the wage margin intensively to increase their hiring. Through the lens of the model, it must mean that the vacancy-posting cost function has a lot of curvature (i.e., θ is high).

Finally, it is worth pointing out that the addition of a vacancy-posting margin has the additional effect of reducing wage dispersion. Figure 7b plots the (relative) wage distribution in the baseline model and in the endogenous-λ extension. The distribution becomes visibly less spread out, and the variance of log wages across workers declines from 0.15 to 0.12, putting it more in line with the empirical target of 0.07 (see "variance of firm effects" in Table VII).

5.4. *Pareto Distribution*

As a final robustness check, I now consider an alternative calibration strategy where the productivity distribution $\Gamma_0(z)$ is Pareto as assumed in Proposition 6. To ensure that the calibration strategy is fully consistent with the baseline model, I parametrize the Pareto distribution so that its mean is normalized to one. In particular, I assume that

$$\Gamma(z) = 1 - (1-\sigma)^{-\frac{1}{\sigma}} z^{-\frac{1}{\sigma}} \quad \text{for } z \geq 1 - \sigma,$$

where the tail index satisfies $\sigma < \frac{1}{1-\alpha}$ to ensure finite aggregate output (see Assumption 1).

I solve the model and calibrate it using the same algorithm as for the baseline model. In Appendix C.4, I report the targeted moment fit. It is worth noting that the Pareto model fails to match the (nontargeted) joint distribution of labor productivity and labor share. In particular, it generates a labor share for firms in the top decile of labor productivity that is much lower than in the data (0.23 versus 0.53, see Appendix Figure E.1e). Relatedly, it implies a low aggregate labor share (0.39 versus 0.62 in the baseline model, see Table XI).

Repeating the model experiment, I find that the response to the productivity dispersion shock is much stronger than in the baseline model (i.e., -2.2% versus -1.3%; see Table Table XI). Shutting down the equilibrium response of the unemployment rate (i.e., row "Pareto distribution + endogenous-b extension" of Table XI), the response is even larger, at -2.4%.

6. CONCLUDING REMARKS

In this paper, I study the effect of labor market imperfections on the labor share in a tractable firm dynamics model with search frictions. In the model, heterogeneous firms grow by accumulating workers and compete through wages in a frictional labor market. In equilibrium, high-productivity firms pay wages below the marginal product while low-productivity firms pay wages *above* the marginal product. I calibrate the model using administrative data covering the universe of firms in Canada over the 2000–2015 period and show that it can replicate a number of nontargeted features of the data, despite being very parsimonious.

The key theoretical finding is that the *distribution* of firm productivity is a central determinant of the aggregate labor share. I find that productivity dispersion effectively weakens the intensity of wage competition in the labor market. Quantitatively, the model predicts that a rise in productivity dispersion of the same magnitude as what the U.S. economy has experienced over the 2000–2015 period causes a 1.3 percentage point decline in the aggregate labor share. Importantly, the decline of the labor share operates via a reallocation of value-added toward firms with a low (and declining) labor share, thus providing a new interpretation to recent firm-level evidence on the decline of the U.S. labor share (see Autor et al. (2020); Kehrig and Vincent (2021)). Exploiting heterogeneity in labor share trends across countries and industries, I provide regression evidence in support of the idea that countries and industries with high (rising) productivity dispersion systematically have a low (declining) labor share.

The analysis in this paper has focused on firm heterogeneity while abstracting from worker heterogeneity. Future work should attempt to quantify which workers benefit the most from working at high-productivity firms. For instance, are "superstar workers" able to extract their full marginal product when working at "superstar firms"?

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Econometrica Supplementary Material

SUPPLEMENT TO "PRODUCTIVITY DISPERSION, BETWEEN-FIRM COMPETITION, AND THE LABOR SHARE"

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APPENDIX A: THEORY

A.1. *Proof of Proposition 1*

FIRST, I FOCUS ON THE PROBLEM of unemployed workers. Their reservation wage $\underline{w}$ is obtained as the solution to $W(\underline{w}) = U$. Equating (3) and (4), I obtain

$$\begin{aligned} b + \mu \int e(z)\left|W\big(w(z)\big) - U\right|_+ \mathrm{d}\Gamma_0(z) + \lambda(1-u) \int \left|W\big(w(z)\big) - U\right|_+ \mathrm{d}P(z) \\ = \underline{w} + \chi \int \big(1 - x(z)\big)\big(W\big(w(z)\big) - U\big)\, \mathrm{d}\Gamma_0(z) \\ + \lambda(1-u) \int \left|W\big(w(z)\big) - U\right|_+ \mathrm{d}P(z). \end{aligned}$$

Using the fact that an employed worker can quit to unemployment, we have $W(w) \geq U$. I remove the max operators and obtain

$$b + \mu \int e(z)\big(W\big(w(z)\big) - U\big)\, \mathrm{d}\Gamma_0(z) = \underline{w} + \chi \int \big(1 - x(z)\big)\big(W\big(w(z)\big) - U\big)\, \mathrm{d}\Gamma_0(z).$$

Rearranging, we obtain

$$\underline{w} = b + \int \big(\mu e(z) - \chi\big(1 - x(z)\big)\big)\big(W\big(w(z)\big) - U\big)\, \mathrm{d}\Gamma_0(z).$$

Second, I focus on the problem of employed workers. I now prove that employed workers accept a job offer w' whenever $w' > w$. To show that $w' > w \implies W(w') > W(w)$, I proceed by contradiction. Set $w' > w$ and assume that $W(w') \leq W(w)$. Using (4), I obtain

$$\begin{aligned} & r\big(W\big(w'\big) - W(w)\big) \\ & = w' - w - \left(\chi \int \big(1 - x(z)\big)\, \mathrm{d}\Gamma_0(z) + \chi \int x(z)\, \mathrm{d}\Gamma_0(z) + \delta\right)\big(W\big(w'\big) - W(w)\big) \\ & \quad + \lambda(1-u) \int \left[\left|W\big(w(z)\big) - W\big(w'\big)\right|_+ - \left|W\big(w(z)\big) - W(w)\right|_+\right] \mathrm{d}P(z). \end{aligned}$$

Using $W(w') \leq W(w)$, we have that

$$\left|W\big(w(z)\big) - W\big(w'\big)\right|_+ \geq \left|W\big(w(z)\big) - W(w)\right|_+,$$

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which implies that

$$
\begin{aligned}
&r\big(W(w') - W(w)\big) \\
&\quad \geq w' - w - \left(\chi \int \big(1 - x(z)\big)\, \mathrm{d}\Gamma_0(z) + \chi \int x(z)\, \mathrm{d}\Gamma_0(z) + \delta\right)\big(W(w') - W(w)\big).
\end{aligned}
$$

Rearranging, we obtain

$$
(r + \chi + \delta)\big(W(w') - W(w)\big) \geq w' - w,
$$

which is a contradiction. Therefore, it must be that $w' > w \implies W(w') > W(w)$.

A.2. *Proof of Lemma 1*

I now provide a derivation of the employment growth function over the range $[\underline{w}, +\infty)$:

$$
\widetilde{g}(w) \equiv \underbrace{\lambda u + \lambda(1-u)\widetilde{P}(w)}_{\text{hiring rate}} - \underbrace{\lambda(1-u)\big(1 - \widetilde{P}(w)\big) - \delta}_{\text{separation rate}}.
$$

The hiring rate is determined by the fraction of job offers that are accepted. A fraction u of meetings are with unemployed workers (who accept all job offers above $\underline{w}$), so the hiring rate out of unemployment is λu. The remainder of meetings are with employed workers who require a wage increase to quit. The measure of employed workers working at firms paying less than w is equal to $\widetilde{P}(w)$, so the hiring rate from employment is given by $\lambda(1-u)\widetilde{P}(w)$.

Separations occur due to quits as well as exogenous job destruction. At rate $\lambda(1-u)$, a worker receives a competing job offer. Since firms meet workers proportionally to their size, the distribution of offered wages is precisely the wage distribution $\widetilde{P}(w)$. The rate at which a firm paying w loses a worker to a competitor is thus given by $\lambda(1-u)(1-\widetilde{P}(w))$. In addition, firms lose workers exogenously at rate δ.

Since the wage distribution $\widetilde{P}$ is a CDF, we have that $\widetilde{P}(w)$ is weakly increasing in w and $\lim_{w\to\infty} \widetilde{P}(w) = 1$. Since $\lambda(1-u) > 0$, it follows that $\widetilde{g}(w)$ is weakly increasing and bounded from above by $\lambda - \delta$.

A.3. *Proof of Proposition 2*

The derivation of $k(z)$ and $y(z)$ follow directly from the first-order condition for capital, so I focus on the derivation of $e(z)$, $x(z)$, and $w(z)$.

Entry and Exit Functions

Firms enter whenever $v(z) > 0$ and exit whenever $v(z) = 0$. I now show that there exists a unique $\underline{z} \geq 0$ such that

$$
z < \underline{z} \quad \implies \quad v(z) = 0, \qquad z > \underline{z} \quad \implies \quad v(z) > 0.
$$

Define $\mathcal{Z}_+ \equiv \{z : v(z) > 0\}$ and $\mathcal{Z}_- \equiv \{z : v(z) = 0\}$. Notice that $\mathcal{Z}_- \cup \mathcal{Z}_+ = [0, +\infty)$. First, I show that $z \in \mathcal{Z}_+ \implies v'(z) > 0$. From (12), we have that

$$
\begin{aligned}
&v(z) > 0 \\
&\implies \quad rv(z) = \max_{w,k}\big\{zk^\alpha - w - Rk + v(z)\widetilde{g}(w)\big\} + \chi\left(\int v(x)\Gamma_0(dx) - v(z)\right).
\end{aligned}
$$

Using the envelope theorem, we have that when $v(z) > 0$,

$$rv'(z) = k^\alpha(z) + v'(z)g(z) - \chi v(z) \quad \Longrightarrow \quad v'(z) = \frac{k(z)^\alpha}{r + \chi - g(z)}.$$

Assumption 2 combined with Lemma 1 implies that $r + \chi - g(z) > 0$ for all $r > 0$, so we obtain the desired result that $v'(z) > 0$. Firms operate whenever $z \in \mathcal{Z}_+$ and $v(z)$ is continuous over $\mathcal{Z}_+$ (differentiability implies continuity), so we have that for all $z \in \inf \mathcal{Z}_+$, $v(z) = 0$. Therefore, the entry/exit threshold is $\underline{z} \equiv \inf \mathcal{Z}_+$.

Wage Function

As is standard in such proofs, there are two cases to consider: a continuous wage distribution and a wage distribution with a point mass. When the wage distribution is continuous, the function $\widetilde{g}(w)$ is differentiable. From the first-order condition (equation (14)), we have that

$$1 = v(z)\widetilde{g}'\big(w(z)\big).$$

Using the definition $g(z) \equiv \widetilde{g}(w(z))$, it follows from the chain rule that $g'(z) = \widetilde{g}'(w(z))w'(z)$. Plugging back in the first-order condition, we obtain

$$w'(z) = v(z)g'(z).$$

To solve for $w(z)$, I use the fact that the marginal firm (i.e., $z = \underline{z}$) is constrained by the worker's reservation wage $\underline{w}$. Without the constraint $w \geq \underline{w}$, firms would never exit as they would be able to achieve zero flow profit by setting $w = 0$ and $k = 0$. I use this insight to obtain the boundary condition $w(\underline{z}) = \underline{w}$. Solving the ODE forward, I obtain the desired result,

$$w(z) = \underline{w} + \int_{\underline{z}}^{z} v(x)g'(x)\,\mathrm{d}x.$$

I now verify that the second-order condition $v(z)\widetilde{g}''(w(z)) < 0$ holds for all $z > \underline{z}$. Since $v(z) > 0$, we only need to verify that $\widetilde{g}''(w(z)) \leq 0$. From the definition of $g(z)$, we have that

$$g'(z) = \widetilde{g}'\big(w(z)\big)w'(z) \quad \Longrightarrow \quad g''(z) = \widetilde{g}''\big(w(z)\big)\big(w'(z)\big)^2 + \widetilde{g}'\big(w(z)\big)w''(z).$$

From the first-order condition, we have that $w'(z) = g'(z)v(z) \Longrightarrow w''(z) = g''(z)v(z) + g'(z)v'(z)$. Putting together,

$$\begin{aligned} g''(z) &= \widetilde{g}''\big(w(z)\big)\big(w'(z)\big)^2 + \widetilde{g}'\big(w(z)\big)\big(g''(z)v(z) + g'(z)v'(z)\big) \\ \Longrightarrow \quad & g''(z)\big(1 - \widetilde{g}'\big(w(z)\big)v(z)\big) = \widetilde{g}''\big(w(z)\big)\big(w'(z)\big)^2 + \widetilde{g}'\big(w(z)\big)g'(z)v'(z). \end{aligned}$$

But from the first-order condition, we have that $1 - \widetilde{g}'(w(z))v(z) = 0$, so

$$\widetilde{g}''\big(w(z)\big) = -\frac{\widetilde{g}'\big(w(z)\big)g'(z)v'(z)}{\big(w'(z)\big)^2} = -\frac{\big(g'(z)\big)^2 v'(z)}{w'(z)}.$$

Since $v'(z) > 0$ for active firms, we have that $\widetilde{g}''(w(z)) > 0 \iff w'(z) > 0$. The case $w'(z) < 0$ can be ruled out as it would imply that $w(z) < \underline{w}$ for all $z > \underline{z}$ so it must be that $w'(z) \geq 0$. Since I am now considering the case with a continuous wage distribution, $w'(z) = 0$ is not possible as it would imply a point-mass wage distribution. Therefore, we have that the second-order condition is satisfied.

Finally, I use a standard argument (see, e.g., Burdett and Mortensen (1998); Coles and Mortensen (2016)) to rule out the possibility of a point mass. Imagine that all firms offer $w(z) = \underline{w}$ so that the wage distribution $\widetilde{P}(w)$ is given by a point mass. A firm could thus deviate by offering $\underline{w} + \epsilon$ (where $\epsilon > 0$), and increase its growth rate discontinuously from $\widetilde{g}(\underline{w}) = \lambda u - \delta$ to $\widetilde{g}(\underline{w}) = \lambda - \delta$ (see equation (8)). An infinitesimal change in the wage offered would then lead to an increase in the continuation value of a firm (see equation (9)) and the equilibrium would unravel. Therefore, the point mass wage distribution cannot be sustained in equilibrium

One-Shot Deviations

In order for the equilbrium to be Markov perfect, there needs to be no profitable one-shot deviation. What prevents a firm from deviating from $w = w(z)$ to $w = \underline{w}$ over a short time interval $[t, t + \mathrm{d}t)$ and then reverting to $w = w(z)$? Workers do not observe productivity directly, so when a firm lowers its wage, workers interpret it as arising due to a negative productivity shock, and expect the wage change to be persistent (productivity shocks are persistent). As soon as the firm would lower the wage, the growth rate of employment would decrease. The wage of the deviating firm over $[t, t + \mathrm{d}t)$ would no longer solve the first-order condition (14) and, therefore, would lead to a lower expected discounted profits.

A.4. *Derivation of Equation (22)*

I now provide a derivation of each term of the *Kolmogorov forward equation* (22) for the employment-weighted productivity distribution $P(z)$:

$$\dot{P}(z) = \underbrace{(1-u)\lambda P(z)\big(P(z)-1\big)}_{\text{Job-to-job flows}} + \underbrace{\frac{u}{1-u}\mu_e \Gamma(z) + u\lambda P(z)}_{\text{Employment inflows}} - \underbrace{(\chi_x + \delta)P(z)}_{\substack{\text{Employment}\\ \text{outflows}}} + \underbrace{\chi_s\big(\Gamma(z) - P(z)\big)}_{\text{Productivity shocks}}.$$

At Poisson rate $\lambda(1-u)$, an employed worker receives a competing job offer. The distribution of productivity (CDF) of such workers, after they have made their decision whether or not to accept the competing job offer, is $P(z)^2$. The reason is that the distribution of z in the population of workers is precisely the employment-weighted productivity distribution $P(z)$ while the distribution of job offers z' is also $P(z)$, since firms meet worker proportionally to their size. The CDF of $\max\{z, z'\}$ is therefore $P(z)^2$.[22] Using the KFE formula for jump processes, the term that accounts for job-to-job flows is therefore

$$\lambda(1-u)\big(P^2(z) - P(z)\big) = \lambda(1-u)P(z)\big(P(z)-1\big).$$

At Poisson μ_e, an unemployed worker meets a potential firm and enters the workforce with productivity distributed according to $\Gamma(z)$. At rate $\lambda(1-u)$, an unemployed worker

[22]If X and Y are two independent random variables with CDFs F, G, then $\mathbb{P}(\max\{X, Y\} \leq z) = F(z)G(z)$.

meets a firm and enters the workforce with productivity distributed according to $P(z)$. Since the ratio of unemployment to employment is $\frac{u}{1-u}$, the term that accounts for unemployment inflows is

$$\frac{u}{1-u}\big(\mu_e\Gamma(z)+\lambda(1-u)P(z)\big)=\frac{u}{1-u}\mu_e\Gamma(z)+u\lambda P(z).$$

At Poisson $\chi_x+\delta$, a worker with productivity distributed according to $P(z)$ is sent to unemployment due to firm exit or exogenous job destruction so that the term which accounts for employment outflows is $-(\chi_x+\delta)P(z)$. At Poisson rate χ_s, an employed worker's productivity resets to a draw from $\Gamma(z)$, so that the term which accounts for productivity shocks is $\chi_s(\Gamma(z)-P(z))$.

A.5. *Proof of Proposition 3*

I break down the proof into three lemmas. All derivations take $\underline{z}$ as given.

LEMMA 2: *There exists a unique real solution* $u\in[0,1)$ *that solves equation* (21) *with* $\dot{u}=0$.

PROOF: Substituting $\dot{u}=0$ in (21), we have

$$0=(1-u)(\chi_x+\delta)-u\big(\mu_e+\lambda(1-u)\big),$$

which is a quadratic equation of the form $au^2+bu+c=0$ with

$$a=\lambda,\qquad b=-(\lambda+\delta+\chi_x+\mu_e),\qquad c=\chi_x+\delta.$$

First, I verify that the discriminant $\Delta\equiv b^2-4ac$ is positive:

$$\begin{aligned}\Delta&=(\lambda+\delta+\chi_x+\mu_e)^2-4\lambda(\chi_x+\delta)\\&\geq(\lambda+\delta+\mu_e+\chi_x)^2-4\lambda(\chi+\delta)\\&\geq\big(\lambda+\delta+\min\{\chi,\mu\}\big)^2-4\lambda(\chi+\delta).\end{aligned}$$

When $\chi\leq\mu$,

$$\Delta\geq(\lambda+\delta+\chi)^2-4\lambda(\chi+\delta)=(\chi+\delta-\lambda)^2.$$

When $\chi>\mu$,

$$\Delta\geq(\lambda+\delta+\mu)^2-4\lambda(\mu+\delta)=(\mu+\delta-\lambda)^2.$$

Putting together, we have that

$$\Delta\geq\big(\min\{\mu,\chi\}+\delta-\lambda\big)^2>0,$$

where the last inequality follows from Assumption 2. There are therefore two real solutions to (21), which I denote by $u^{(\pm)}$.

$$u^{\pm}=\frac{\lambda+\delta+\chi_x+\mu_e\pm\sqrt{(\lambda+\delta+\chi_x+\mu_e)^2-4\lambda(\chi_x+\delta)}}{2\lambda}.$$

I now use the fact that $\sqrt{\Delta} > \min\{\mu, \chi\} + \delta - \lambda$ to show that $u^+ > 1$:

$$u^+ \geq \frac{\lambda + \delta + \min\{\chi, \mu\} + \sqrt{\Delta}}{2\lambda} > \frac{2\lambda}{2\lambda} = 1.$$

Therefore, u^+ is not the solution (the unemployment rate cannot be greater than one). I now show that u^- is the solution (i.e., it satisfies $0 \leq u^- < 1$). First, we have that

$$u^- \leq \frac{\lambda + \delta + \max\{\mu, \chi\}}{2\lambda} < 1,$$

where the last equality follows from

$$\lambda + \delta + \max\{\mu, \chi\} < 2\lambda \quad \Longleftrightarrow \quad \delta + \max\{\mu, \chi\} - \lambda < 0,$$

which is true under Assumption 2. Finally, we have that $u^{(-)} \geq 0$, since

$$b < \sqrt{b^2 - 4ac} \quad \Longleftrightarrow \quad 4ac \geq 0$$

and $4ac = \lambda(\delta + \chi_x)\delta \geq 0$. *Q.E.D.*

LEMMA 3: *The stationary unemployment rate satisfies* $u < \frac{\chi+\delta}{\lambda}$.

PROOF: Using the fact the discriminant is positive, we have that

$$\Delta > 0 \quad \Longrightarrow \quad u < \frac{\lambda + \min\{\mu, \chi\} + \delta}{2\lambda}.$$

Under Assumption 2, we have that $\lambda < \min\{\mu, \chi\} + \delta$, so that

$$u < \frac{\min\{\mu, \chi\} + \delta}{\lambda} \leq \frac{\chi + \delta}{\lambda}.$$ *Q.E.D.*

LEMMA 4: *There exists a unique function P that is a valid CDF and solves* (22) *with* $\dot{P}(z) = 0$.

PROOF: Setting $\dot{P}(z) = 0$ in (22), I obtain a quadratic equation in $P(z)$ of the form $aP^2(z) + bP(z) + c(z) = 0$, with coefficients given by

$$\begin{aligned} a &= (1-u)\lambda, \\ b &= -\big((1-u)\lambda + \chi_x + \chi_s + \delta - u\lambda\big), \\ c(z) &= \left(\frac{u}{1-u}\mu_e + \chi_s\right)\Gamma(z), \end{aligned}$$

Notice that, when $\dot{u} = 0$, we have from (22) that

$$\frac{u}{1-u}\mu_e = \chi_x + \delta - u\lambda.$$

Moreover, $\chi_x + \chi_s = \chi$, so the coefficients can be simplified to

$$a = (1-u)\lambda, \qquad b = -\big((1-u)\lambda + \chi + \delta - u\lambda\big), \qquad c(z) = (\chi + \delta - u\lambda)\Gamma(z).$$

I now show that the determinant

$$\Delta(z) \equiv \big(\lambda(1-u)+\chi+\delta-\lambda u\big)^2 - 4(1-u)\lambda(\chi+\delta-u\lambda)\Gamma(z),$$

is positive for all z. First, I show that $\Delta'(z) < 0$. Since $\Gamma'(z) > 0$, the statement is implied by $\chi+\delta-u\lambda > 0$, which is true (see Lemma 3). Since $\Delta(z)$ is decreasing, it sufficient to show that it is positive in the limit:

$$\lim_{z\to\infty} \Delta(z) = \big(\lambda(1-u)+\chi+\delta-\lambda u\big)^2 - 4(1-u)\lambda(\chi+\delta-u\lambda) = (\chi+\delta-\lambda)^2 > 0,$$

where the last inequality follows from Assumption 2. We therefore have two real solutions given by

$$\begin{aligned} P^{\pm}(z) = \big(&\lambda(1-u)+\chi+\delta-u\lambda \\ &\pm\sqrt{\big(\lambda(1-u)+\chi+\delta-\lambda u\big)^2 - 4\lambda(1-u)(\chi+\delta-u\lambda)\Gamma(z)}\big) \\ &/\big(2\lambda(1-u)\big). \end{aligned}$$

First, I show that $P^+(z) > 1$ for all $z > \underline{z}$, which implies that P^+ is not a valid CDF. Using the fact that $\sqrt{\Delta(z)} \geq \lambda(1-u)+\chi+\delta-u\lambda$, we have that

$$P^+(z) \geq \frac{\lambda(1-u)+\chi+\delta-u\lambda}{\lambda(1-u)} > 1 \quad\iff\quad \chi+\delta-u\lambda > 0,$$

and the last inequality follows from Lemma 3. I now show that P^- is a valid CDF, meaning that (i) $(P^-)'(z) \geq 0$, (ii) $P^-(\underline{z}) = 0$, and (iii) $\lim_{z\to\infty} P^-(z) = 1$. First,

$$\big(P^-\big)'(z) = -\frac{\Delta'(z)}{\sqrt{\Delta(z)}4a} > 0$$

The last equality follows from $\Delta'(z) < 0$ (already shown). Second,

$$P^-(\underline{z}) = \frac{-b-|b|}{2a} = 0,$$

where the last equality follows from $b < 0$. Finally, I have already shown that $\lim \Delta(z) = (\chi+\delta-\lambda)^2$, which implies, under Assumption 2, that $\lim \sqrt{\Delta(z)} = \chi+\delta-\lambda$. We therefore have that

$$\lim_{z\to\infty} P^-(z) = \frac{\lambda(1-u)+\chi+\delta-u\lambda-(\chi+\delta-\lambda)}{2\lambda(1-u)} = \frac{2\lambda(1-u)}{2\lambda(1-u)} = 1,$$

which concludes the proof. *Q.E.D.*

A.6. *Proof of Proposition 4*

The fact that $\mathrm{LS}(\underline{z}) > 1-\alpha$ follows directly from (28):

$$\underline{w} = (1-\alpha)y(\underline{z}) + \chi\int_0^\infty v(x)\,\mathrm{d}\Gamma_0(x) \quad\implies\quad \mathrm{LS}(\underline{z}) = 1-\alpha+\frac{\chi\displaystyle\int_0^\infty v(x)\,\mathrm{d}\Gamma_0(x)}{y(\underline{z})}.$$

Since $v(\underline{z}) = 0$ and $v'(z) > 0$, we have that $\int_0^\infty v(x)\,\mathrm{d}\Gamma_0(x) > 0$. By continuity of $\mathrm{LS}(z)$, we have that there exist a z' such that for all $z < z'$ we have that $\mathrm{LS}(z) > 1 - \alpha$. To show that there exists a z'' such that for all $z > z''$, $\mathrm{LS}'(z) < 0$, first notice that

$$\mathrm{LS}'(z) < 0 \quad \Longleftrightarrow \quad \frac{w'(z)}{w(z)} < \frac{y'(z)}{y(z)} \quad \Longleftrightarrow \quad w'(z)z < \frac{1}{1-\alpha} w(z).$$

To show that there exists a z'' such that this inequality holds for all $z > z''$, I first prove three lemmas.

LEMMA 5: *The value function is bounded from above by*

$$\overline{v}(z) = -\frac{1-\alpha}{r+\chi+\delta-\lambda} y(\underline{z}) + \frac{1-\alpha}{r+\chi+\delta-\lambda} y(z).$$

PROOF: Since $w(z) > \underline{w}$ and $g(z) < \lambda - \delta$ (Lemma 1), then we can construct that upper bound $\overline{v}(z)$ by plugging $w(z) = \underline{w}$ (see equation (28) for equilibrium expression for $\underline{w}$) and $g(z) = \lambda - \delta$ in the HJB (9).

$$r\overline{v}(z) = (1-\alpha)\big(y(z) - y(\underline{z})\big) + (\lambda - \delta - \chi)\overline{v}(z).$$

Solving for $\overline{v}(z)$, we obtain the desired result. Under Assumption 2, we have that $r + \chi + \delta - \lambda > 0$. *Q.E.D.*

LEMMA 6: *The employment weighted productivity distribution is asymptotically equivalent to* $\Gamma(z)$,

$$P'(z) \sim \frac{\chi + \delta - u\lambda}{\chi + \delta - \lambda} \Gamma'(z)$$

PROOF: From Lemma 8 below, we have that $P'(z) = \frac{\chi+\delta-u\lambda}{\chi - g(z)}\Gamma'(z)$. From Lemma 1, we have that $g(z) \to \lambda - \delta$. *Q.E.D.*

LEMMA 7: *The wage function* $w(z)$ *is bounded from above by* $\overline{w} < \infty$.

PROOF: Let $\overline{w} \equiv \lim_{z\to\infty} w(z) = \underline{w} + \int_{\underline{z}}^\infty v(z) g'(z)\,\mathrm{d}z$. Using the upper bound for $\overline{v}(z)$, we have that $\overline{w} \leq \underline{w} + (1-\alpha)\frac{2\lambda(1-u)}{r+\chi+\delta-\lambda}\int_{\underline{z}}^\infty y(z)\,\mathrm{d}P(z) < \infty$. Notice that the last term is aggregate output, which is finite under Assumption 1. Moreover, Assumption 2 implies that $r + \chi + \delta - \lambda > 0$. *Q.E.D.*

Equipped with these results, I now show that there exists a z'' such that for all $z > z''$,

$$w'(z)z < \frac{1}{1-\alpha} w(z).$$

By continuity, I only need to show that the right-hand size converges to a positive number and that the left-hand side converges to zero. The right-hand side converges to $\frac{1}{1-\alpha}\overline{w} > 0$. The left-hand side is bounded from below by 0 (since $w'(z) > 0$) and is bounded from above by

$$2\lambda(1-u)\overline{v}(z)P'(z) \sim \frac{2\lambda(1-u)}{r+\chi+\delta-\lambda} z^{1+\frac{1}{1-\alpha}}\Gamma'(z),$$

where the asymptotic equivalence relationship uses the two lemmas combined with $w'(z) = 2\lambda(1-u)P'(z)$. Under Assumption 1, we have that $z^{1+\frac{1}{1-\alpha}}\Gamma'(z) \to 0$, which concludes the proof.

A.7. *Proof of Proposition 5*

The aggregate labor share is the ratio of aggregate wages to aggregate output. Rearranging, I obtain

$$\mathrm{LS} = \frac{(1-u)\int_{\underline{z}}^{\infty} w(z)\,\mathrm{d}P(z)}{(1-u)\int_{\underline{z}}^{\infty} y(z)\,\mathrm{d}P(z)} = \frac{\int_{\underline{z}}^{\infty} w(z)\,\mathrm{d}P(z)}{Y} = \int_{\underline{z}}^{\infty} \mathrm{LS}(z)\frac{y(z)}{Y}\,\mathrm{d}P(z).$$

To obtain equation (29), I first prove the following lemma

LEMMA 8: *The relationship between the employment-weighted productivity distribution $P(z)$ and the productivity distribution $\Gamma(z)$ is given by*

$$\mathrm{d}P(z) = \frac{\chi + \delta - \lambda u}{\chi - g(z)}\,\mathrm{d}\Gamma(z). \tag{44}$$

PROOF: Using (24), I now obtain an expression for $P'(z)$,

$$\begin{aligned} P'(z) &= \frac{\chi + \delta - u\lambda}{\sqrt{\big(\lambda(1-u) + \chi + \delta - \lambda u\big)^2 - 4\lambda(1-u)(\chi + \delta - u\lambda)\Gamma(z)}}\Gamma'(z), \\ &= \frac{\chi + \delta - u\lambda}{\lambda(1-u) + \chi + \delta - u\lambda - 2\lambda(1-u)P(z)}\Gamma'(z), \\ &= \frac{\chi + \delta - u\lambda}{\chi - g(z)}\Gamma'(z), \end{aligned}$$

where the third equation uses the definition of $g(z)$ (equation (18)). Rearranging, I obtain the desired result:

$$\frac{\mathrm{d}P(z)}{\mathrm{d}\Gamma(z)} = \frac{\chi + \delta - u\lambda}{\chi - g(z)}. \qquad \textit{Q.E.D.}$$

Notice that the term $\lambda u - \delta$ is equal to the average growth rate of existing firms $\overline{g} \equiv \int_{\underline{z}}^{\infty} g(z)\,\mathrm{d}P(z)$:

$$\begin{aligned} \int_{\underline{z}}^{\infty} g(z)\,\mathrm{d}P(z) &= \lambda u - \delta + \int_{\underline{z}}^{\infty} \big(\lambda(1-u)P(z) - \lambda(1-u)\big(1 - P(z)\big)\big)\,\mathrm{d}P(z), \\ &= \lambda u - \delta + \lambda(1-u)\frac{1}{2} - \lambda(1-u)\frac{1}{2}, \\ &= \lambda u - \delta. \end{aligned}$$

Substituting $\mathrm{d}P(z) = \frac{\chi-\overline{g}}{\chi-g(z)}\,\mathrm{d}\Gamma(z)$ in the expression for the labor share, I obtain the desired result:

$$\mathrm{LS} = \int_{\underline{z}}^{\infty} \mathrm{LS}(z) \times \frac{\chi-\overline{g}}{\chi-g(z)}\frac{y(z)}{Y} \times \mathrm{d}\Gamma(z).$$

A.8. *Firm Size Distribution*

The law of motion for the measure of firms F is

$$\dot{F} = u\mu_e - \chi_x F.$$

In a stationary equilibrium, we have that $\dot{F} = 0$, which implies that the firm entry rate $\frac{u\mu_e}{F}$ must equal the firm exit rate χ_x. The Kolmogorov forward equation for the joint distribution of productivity and size $\varphi(z, N) : [\underline{z}, +\infty] \times \mathbb{R}_+ \to \mathbb{R}_+$ across firms is given by

$$\dot{\varphi}(z, N) = \underbrace{\chi_s\left(\gamma(z)\int_{\underline{z}}^{\infty} \varphi(x, N)\,\mathrm{d}x - \varphi(z, N)\right)}_{\text{productivity shocks}} - \underbrace{g(z)\frac{\partial}{\partial N}\big(\varphi(z, N)N\big)}_{\text{firm growth}} \tag{45}$$

$$+ \underbrace{\mu_e u\gamma(z)\psi(N-1) - \chi_x\varphi(z, N)}_{\text{firm turnover}}, \tag{46}$$

where $\psi(N)$ is the Dirac delta and $\gamma(z) = \Gamma'(z)$. I now characterize the upper tail of the firm-size distribution.

PROPOSITION 1: *The upper tail of the firm size distribution obeys a power law*

$$\mathbb{P}(N > n) \sim Cn^{-\zeta}, \tag{47}$$

for some constant $C > 0$ *and Pareto exponent* $\zeta > 1$.

PROOF: Suppose that the solution (i.e., $\dot{\varphi}(z, N) = 0$) satisfies $\varphi(z, N) \sim \overline{\varphi}(z)N^{-(1+\zeta)}$, for some positive function $\overline{\varphi}$. Substituting the asymptotic conjecture in KFE, I obtain

$$\begin{aligned}
0 &= \chi_s\left(\gamma(z)\int \overline{\varphi}(z')\,\mathrm{d}z' - \overline{\varphi}(z)\right)N^{-(1+\zeta)} + \zeta g(z)\overline{\varphi}(z)N^{-(1+\zeta)} - \chi_x\overline{\varphi}(z)N^{-(1+\zeta)} \\
&= \chi_s\gamma(z)\int \overline{\varphi}(z') + \zeta g(z)\overline{\varphi}(z) - \chi\overline{\varphi}(z) \\
&= \underbrace{(\chi_s\mathcal{D}_\gamma\mathcal{K} + \zeta\mathcal{D}_g - \chi\mathcal{I})}_{\equiv\mathcal{A}(\zeta)}\overline{\varphi}(z).
\end{aligned}$$

For any functions (f, g), I define the operators $\mathcal{D}_\cdot, \mathcal{K}, \mathcal{I}$ by the following actions $\mathcal{D}_f g(z) \equiv f(z)g(z)$, $\mathcal{K}f(z) \equiv \int f(z')\,\mathrm{d}z'$, and $\mathcal{I}f(z) \equiv f(z)$. The Pareto exponent ζ and function $\overline{\varphi}(z)$ are thus the solution to the following equation:

$$\mathcal{A}(\zeta)\overline{\varphi}(z) = 0.$$

See Beare and Toda (2022) and Beare, Seo, and Toda (2021) for a formal treatment. *Q.E.D.*

To compute ζ, I discretize the operator $\mathcal{A}(\zeta)$ (for a guess of ζ) and compute the largest eigenvalue. The solution ζ is such that the largest eigenvalue is zero.

A.9. *Proof of Proposition 6*

I now prove three lemmas that I will use for the proof of Proposition 6.

LEMMA 9: *For all $z > \underline{z}$, the derivative of the firm value function is given by*

$$v'(z) = \frac{\frac{1}{z}y(z)}{r + \chi - g(z)}.$$

PROOF: Using the HJB for continuing firms (9) combined with the envelope theorem, I obtain

$$rv'(z) = (1 - \alpha)y'(z) + g(z)v'(z) - \chi v'(z).$$

Solving for $v'(z)$, I obtain the desired result

$$v'(z) = \frac{(1-\alpha)y'(z)}{r + \chi - g(z)} = \frac{1}{z}\frac{y(z)}{r + \chi - g(z)},$$

where the last equality uses the definition of labor productivity $y(z)$. *Q.E.D.*

LEMMA 10: *Suppose that the assumptions of Proposition* 6 *are satisfied. For all $z > \underline{z}$, the CDF of the productivity distribution of active firms satisfies*

$$\sigma \, \mathrm{d}\Gamma(z) = \frac{1}{z}\big(1 - \Gamma(z)\big) \, \mathrm{d}z.$$

PROOF:

$$\mathrm{d}\Gamma(z) = \sigma z^{-(1+\frac{1}{\sigma})} \, \mathrm{d}z \quad \Longrightarrow \quad \sigma z \, \mathrm{d}\Gamma(z) = z^{-\frac{1}{\sigma}} \, \mathrm{d}z = \big(1 - \Gamma(z)\big) \, \mathrm{d}z. \qquad \textit{Q.E.D.}$$

LEMMA 11: *Suppose that the assumptions of Proposition* 6 *are satisfied. The expected value of a new firm is given by*

$$\int v(x) \, \mathrm{d}\Gamma(x) = \underline{v} + \frac{\sigma}{\chi - \overline{g}} Y,$$

where Y is aggregate output.

PROOF:

$$\begin{aligned}
\int v(x) \, \mathrm{d}\Gamma(x) &= \underline{v} + \int_{\underline{z}}^{\infty} \int_{\underline{z}}^{x} v'(s) \, \mathrm{d}s \, \mathrm{d}\Gamma(z) \\
&= \underline{v} + \int_{\underline{z}}^{\infty} \left(\int_{\underline{z}}^{x} \frac{1}{s}\frac{y(s)}{\chi - g(s)} \, \mathrm{d}s \right) \mathrm{d}\Gamma(x) \\
&= \underline{v} + \int_{\underline{z}}^{\infty} \frac{1}{s}\frac{y(s)}{\chi - g(s)} \left(\int_{s}^{\infty} \mathrm{d}\Gamma(x) \right) \mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \underline{v} + \int_{\underline{z}}^{\infty} \frac{y(s)}{\chi - g(s)} \frac{1}{s}\big(1 - \Gamma(s)\big)\,\mathrm{d}s \\
&= \underline{v} + \frac{\sigma}{\chi - \overline{g}} \int_{\underline{z}}^{\infty} y(s) \frac{\chi - \overline{g}}{\chi - g(s)}\,\mathrm{d}\Gamma(s) \\
&= \underline{v} + \frac{\sigma}{\chi - \overline{g}} \int_{\underline{z}}^{\infty} y(s)\,\mathrm{d}P(s).
\end{aligned}
$$

The second equality uses Lemma 9 and the fifth one uses Lemma 10. *Q.E.D.*

I now derive an expression for aggregate profits $\int \pi(z)\,\mathrm{d}P(z)$, where $\pi(z) \equiv (1 - \alpha)y(z) - w(z)$. Using the HJB (9) combined with $r = 0$, we have

$$
\pi(z) = \big(\chi - g(z)\big)v(z) - \chi_s \int v(x)\,\mathrm{d}\Gamma(x).
$$

Integrating against the density $\mathrm{d}P(z)$, we obtain

$$
\begin{aligned}
\int_{\underline{z}}^{\infty} \pi(z)\,\mathrm{d}P(z) &= \int_{\underline{z}}^{\infty} v(x)\big(\chi - g(x)\big)\,\mathrm{d}P(x)\,\mathrm{d}x - \chi_s \int v(x)\,\mathrm{d}\Gamma(z) \\
&= (\chi - \overline{g}) \int_{\underline{z}}^{\infty} v(x)\,\mathrm{d}\Gamma(z) - \chi_s \int v(x)\,\mathrm{d}\Gamma(z) \\
&= (\chi_x - \overline{g}) \int_{\underline{z}}^{\infty} v(x)\,\mathrm{d}\Gamma(z) \\
&= (\chi_x - \overline{g}) \left(\underline{v} + \frac{\sigma}{\chi - \overline{g}} \int_{\underline{z}}^{\infty} y(s)\,\mathrm{d}P(s) \right).
\end{aligned}
$$

The second equation uses Lemma 8, the third one uses the fact that $\chi = \chi_s + \chi_x$, and the fourth uses Lemma 11. Denoting aggregate output and profits as $Y \equiv \int_{\underline{z}}^{\infty} y(s)\,\mathrm{d}P(s)$ and $\Pi \equiv \int_{\underline{z}}^{\infty} \pi(s)\,\mathrm{d}P(s)$, we have that

$$
\Pi = (\chi_x - \overline{g}) \left(\underline{v} + \frac{\sigma}{\chi - \overline{g}} Y \right). \tag{48}
$$

Using the fact that $\Pi / Y = 1 - \alpha - \mathrm{LS}$ combined with $\underline{v} = 0$, we obtain

$$
\mathrm{LS} = 1 - \alpha - \frac{\chi_x - \overline{g}}{\chi - \overline{g}} \sigma,
$$

which is the main result. The last thing to prove is that $0 < \frac{\chi_x - \overline{g}}{\chi - \overline{g}} < 1$. Using the law of motion for employment in steady state (i.e., equation (21)), we have that $\chi_x - \overline{g} = \frac{u}{1-u}\mu_e > 0$, which implies that $\frac{\chi_x - \overline{g}}{\chi - \overline{g}} > 0$. Moreover, $\chi_x < \chi$ implies that $\frac{\chi_x - \overline{g}}{\chi - \overline{g}} < 1$.

APPENDIX B: BENCHMARK MODELS AND EXTENSIONS

B.1. *Burdett and Mortensen (1998) With Capital*

I now present a derivation of the model which I refer to as "Burdett–Mortensen" in the main text. It corresponds to an extension of the model in Burdett and Mortensen

(1998) with capital as a factor of production. Unless stated otherwise, functional forms and notations are exactly as in the baseline model.

Main Equations

The laws of motion for the wage distribution $P(w)$ and firm-level employment $N(w)$ are respectively given by

$$\dot{N}(w) = \underbrace{\lambda P(w)}_{\text{Hires}} - \underbrace{\delta N(w)}_{\text{Layoffs}} - \underbrace{\lambda \overline{F}(w) N(w)}_{\text{Quits}},$$

$$\dot{P}(w) = \underbrace{\delta\big(1\{w \geq 0\} - P(w)\big)}_{\text{Job destruction}} + \underbrace{\lambda\big(F(w)P(w) - P(w)\big)}_{\text{Job creation + Churn}},$$

where $F(w)$ denotes the (endogenously-determined) distribution of wage offers and $\overline{F}(w) \equiv 1 - F(w)$. The value $P(0)$ is by convention the unemployment rate (i.e., the measure of workers working at firms that pay $w = 0$). Setting $\dot{N}(w) = 0$ and $\dot{P}(w) = 0$, I obtain

$$N(w) = \frac{\lambda\delta}{\big(\delta + \lambda\overline{F}(w)\big)^2}, \qquad P(w) = \frac{\delta}{\delta + \lambda\overline{F}(w)}.$$

Let $b > 0$ be the flow value of unemployment. As in Burdett and Mortensen (1998), I focus on a long-run steady-state where firms choose a constant wage policy $w \geq b$ and capital stock per worker $k \geq 0$ as to maximize long-run flow profits defined as

$$\begin{aligned} v(z) &= \max_{w,k} \lambda P(w) \frac{zk^{\alpha} - Rk - w}{r + \delta + \lambda\overline{F}(w)} \\ &= \max_{w} \delta\lambda \frac{(1-\alpha)y(z) - w}{\big(r + \delta + \lambda\overline{F}(w)\big)\big(\delta + \lambda\overline{F}(w)\big)}. \end{aligned}$$

Flow profits are given by the measure of hires $\lambda P(w)$ times the present-value of a hire. In the second expression, I use the steady-state expression for $P(w)$ and the optimal capital stock is $k(z) = (\frac{z\alpha}{R})^{\frac{1}{1-\alpha}}$ (i.e., same formula as in the baseline model; see Proposition 2).

Denoting $\widetilde{\phi}(w) \equiv \frac{\delta\lambda}{(r+\delta+\lambda\overline{F}(w))(\delta+\lambda\overline{F}(w))}$, the first-order condition for wages is given by

$$\widetilde{\phi}'\big(w(z)\big)\big((1-\alpha)y(z) - w(z)\big) - \widetilde{\phi}\big(w(z)\big) = 0.$$

Defining $\phi(z) \equiv \widetilde{\phi}(w(z))$, I obtain

$$\begin{aligned} &\phi'(z)\big((1-\alpha)y(z) - w(z)\big) - \phi(z)w'(z) = 0 \\ &\implies \quad (1-\alpha)y(z)\phi'(z) = \big(w(z)\phi(z)\big)'. \end{aligned}$$

This is an ODE. Combined with the initial condition $w(\underline{z}) = b$, I obtain the following solution:

$$\begin{aligned} &w(z)\phi(z) - b\phi(\underline{z}) = (1-\alpha)\phi(\underline{z})y(\underline{z}) - \int_{\underline{z}}^{z} y'(x)\phi(x)\,\mathrm{d}x \\ &\implies \quad w(z) = (1-\alpha)y(z) + \big(b - (1-\alpha)y(\underline{z})\big)\frac{\phi(\underline{z})}{\phi(z)} - (1-\alpha)\int_{\underline{z}}^{z} y'(x)\frac{\phi(x)}{\phi(z)}\,\mathrm{d}x. \end{aligned}$$

The entry threshold $\underline{z}$ satisfies $(1-\alpha)y(\underline{z}) = b$, which ensures that the marginal firm makes zero profit. The formula simplifies to

$$w(z) = (1-\alpha)\left[y(z) - \int_{\underline{z}}^{z} y'(x)\frac{\phi(x)}{\phi(z)}\,\mathrm{d}x\right]. \tag{49}$$

Let Γ_0 be the distribution of productivity of potential entrants and $\Gamma(z) = \frac{\Gamma_0(z)-\Gamma_0(\underline{z})}{1-\Gamma_0(\underline{z})}$ be the truncated distribution where the threshold $\underline{z}$ is given by $\underline{z} = \frac{b^{1-\alpha}R^{\alpha}}{(1-\alpha)^{1-\alpha}\alpha^{\alpha}}$. Using the fact that the wage schedule is increasing in z, we have that the equilibrium is characterized by

$$\begin{aligned} N(z) &= \frac{\lambda\delta}{\left(\delta+\lambda\overline{\Gamma}(z)\right)^2}, \qquad P(z) = \frac{\delta}{\delta+\lambda\overline{\Gamma}(z)}, \\ \phi(z) &= \frac{\delta\lambda}{\left(r+\delta+\lambda\overline{\Gamma}(z)\right)\left(\delta+\lambda\overline{\Gamma}(z)\right)}, \end{aligned} \tag{50}$$

where $P(z) \equiv P(w(z))$.

Pareto Special Case

I now provide a closed-form solution for the aggregate labor share in the Pareto special case with no discounting (i.e., same assumptions as in Proposition 6). First, note that $r = 0$ implies $\phi(z) = N(z)$. Integrating equation (49) against $N(z)\,\mathrm{d}\Gamma(z)$, I obtain

$$\underbrace{\int_b^{\infty} w(x)N(x)\,\mathrm{d}\Gamma(x)}_{wN} = (1-\alpha)\underbrace{\int_b^{\infty} xN(x)\,\mathrm{d}\Gamma(x)}_{Y} - \underbrace{(1-\alpha)\int_b^{\infty}\int_b^{z} y'(s)N(s)\,\mathrm{d}s\,\mathrm{d}\Gamma(x)}_{\Pi},$$

where wN is aggregate worker compensation, Y is aggregate output, and Π is aggregate profits. Using the fact that $(1-\alpha)y'(s) = \frac{1}{s}y(s)$, I obtain

$$\begin{aligned} \Pi &= \int_b^{\infty} \frac{y(s)}{s} N(s)\left(\int_b^{\infty} 1\{s \le x\}\,d\Gamma(x)\right)dy \\ &= \int_b^{\infty} \frac{y(s)}{s} N(s)\Gamma(s)\,ds. \end{aligned}$$

Finally, using the fact that $1-\Gamma(s) = \sigma s\frac{d\Gamma(s)}{ds}$, I obtain

$$\Pi = \sigma\int_b^{\infty} y(s)N(y)\Gamma(y)\,dy = \sigma Y.$$

The labor share $\mathrm{LS} \equiv wN/Y$ is therefore given by

$$\mathrm{LS} = 1-\alpha-\sigma. \tag{51}$$

Calibration

The free parameters of the model are $(\alpha, R, \lambda, \delta, b, \eta)$. First, I calibrate (α, R, η)—which govern the capital share, discount rate, and productivity dispersion—exactly as in the baseline model (see Section 3).

The remaining parameters are (λ, δ). In the baseline model, those parameters are identified using job reallocation rates across firms as well as the unemployment rate. In Burdett–Mortensen's long-run steady state, there is no job reallocation. Instead, I use the unemployment rate and the layoff rate as empirical targets.

I set $\delta = 1 - e^{-12\times 0.004} = 0.0469$ to match a monthly employment-to-unemployment probability of 0.004 as estimated in Nakamura, Nakamura, Phong, and Steinsson (2019) using Canadian microdata. Then I choose λ to match the unemployment target of 7.13% as in the baseline model.

$$u \equiv P(0) = \frac{\delta}{\delta + \lambda} \qquad \Longrightarrow \qquad \lambda = \delta \times \frac{1-u}{u} = 0.6109.$$

Finally, b is chosen such that the rank of the entry threshold $\Gamma_0(\underline{z})$ in the Burdett–Mortensen model coincides with the rank of the entry threshold in the full model. This last step ensures that the distribution of productivity amongst active firms (i.e., $\Gamma(z)$) is exactly the same in the baseline model and the Burdett–Mortensen model.

B.2. *Coles and Mortensen (2016) With Balanced-Matching and Capital*

I now present a derivation of the model which I refer to as "Coles–Mortensen" in the main text. It corresponds to a version of the model in Coles and Mortensen (2016) with balanced matching and capital as a factor of production. Unless stated otherwise, functional forms and notation are exactly as in the baseline model.

In the Coles–Mortensen model, the minimum productivity level $\underline{z}$ is exogenous, which implies that the firm exit rate χ_x and productivity shock rate χ_s are exogenous as well (see equation (20)). Coles and Mortensen (2016) assume that flow profits are nonnegative at every date and state, which implies that

$$\underline{w} \leq (1-\alpha)y(\underline{z}).$$

I consider the most conservative case (i.e., where flow profits of the lowest productivity firms are zero, as in the Burdett–Mortensen). Assumption 3 in the baseline model is thus replaced by

$$\underline{w} = (1-\alpha)y(\underline{z}).$$

Main Equations

Given an exogenous $\underline{z}$ (which pins down χ_x and χ_s), the equilibrium unemployment rate u, employment-weighted productivity distribution $P(z)$ are exactly as in the baseline model (see equations (23) and (24)). This implies that $g(z) = \lambda u + \lambda(1-u)P(z) - \lambda(1-u)P(z) - \delta$, $v'(z) = \frac{1}{z}\frac{y(z)}{r+\chi-g(z)}$, and $v(z) - \underline{v} = \int_{\underline{z}}^{z} v'(x)\,\mathrm{d}x$ are also exactly as in the standard model.

I first derive an expression for $v(\underline{z})$, which I then use to derive an expression for $v(z)$ and $w(z)$. Using the HJB for active firms (9), we have

$$rv(z) = \pi(z) + g(z)v(z) + \chi_s \int v(x)\,\mathrm{d}\Gamma(x) - \chi v(z).$$

Evaluating at $z = \underline{z}$ and using the fact that $\underline{w} = (1-\alpha)y(\underline{z})$, I obtain

$$\begin{aligned}(r+\chi-\underline{g})\underline{v} &= \chi_s \int v(x)\,\mathrm{d}\Gamma(x) \\ &= \chi_s\underline{v} + \chi_s \int \big(v(x)-\underline{v}\big)\,\mathrm{d}\Gamma(x),\end{aligned}$$

where $\underline{g} \equiv g(\underline{z})$. Using the fact that $\chi = \chi_s + \chi_x$, I obtain

$$\underline{v} = \frac{\chi_s}{r+\chi_x-\underline{g}} \int \big(v(x)-\underline{v}\big)\,\mathrm{d}\Gamma(x). \tag{52}$$

From the first-order condition for wages, we have that

$$w'(z) = g(z)v(z) \quad \implies \quad w(z) = \underline{w} + \int_{\underline{z}}^{z} g'(x)v(x).$$

Using the fact that $\underline{w} = (1-\alpha)y(\underline{z})$, we have that

$$\begin{aligned}w(z) &= (1-\alpha)y(\underline{z}) + \int_{\underline{z}}^{z} g'(x)v(x)\,\mathrm{d}x \\ &= (1-\alpha)y(\underline{z}) + \big(g(z)-\underline{g}\big)\underline{v} + \int_{\underline{z}}^{z} g'(x)\big(v(x)-\underline{v}\big)\,\mathrm{d}x.\end{aligned}$$

Combining with (52), I obtain

$$\begin{aligned}w(z) = (1-\alpha)y(\underline{z}) &+ \big(g(z)-\underline{g}\big)\frac{\chi_s}{r+\chi_x-\underline{g}} \int_{\underline{z}}^{\infty} \big(v(x)-\underline{v}\big)\,\mathrm{d}\Gamma(x) \\ &+ \int_{\underline{z}}^{z} g'(x)\big(v(x)-\underline{v}\big)\,\mathrm{d}x. \end{aligned}\tag{53}$$

This equation is useful because it expresses the wage schedule in the Coles–Mortensen model in terms of objects that depend only on the exogenous threshold $\underline{z}$: $(1-\alpha)y(\underline{z})$, $g(z)-\underline{g}$, $\frac{\chi_s}{r+\chi_x-\underline{g}}$, $g'(z)$, and $v(x)-\underline{v}$. Therefore, the Coles–Mortensen model can be solved entirely in closed form.

Note that, if the baseline model and Coles–Mortensen models are calibrated to imply the same productivity threshold $\underline{z}$, then they will imply the same allocation of workers across firms $P(z)$, and only differ in the equilibrium breakdown between wages and profits $w(z)$.

Calibration

The model parameters are $(\underline{z}, \lambda, \mu, \delta, \chi, \eta)$. First, I choose $\underline{z}$ so that the rank of the entry threshold $\Gamma_0(\underline{z})$ in the Coles–Mortensen model coincides with the rank of the entry threshold in the baseline model. This step ensures that the distribution of productivity among active firms $\Gamma(z)$ is exactly the same in the baseline model and the Coles–Mortensen model. Then I set $(\lambda, \mu, \delta, \chi, \eta)$ to the same values as in the baseline model (see Section 3.1). This calibration strategy implies that the Coles–Mortensen model implies the same targeted moments as the baseline model (see Section 3.1). The only difference between the baseline model and Coles–Mortensen is the wage function, which pins down the split between profits and wages.

Pareto Special Case

I now provide a closed-form solution for the aggregate labor share in the Pareto special case with no discounting (i.e., same assumptions as in Proposition 6). Combining Proposition 11 and equation (52), I obtain the following expression:

$$\underline{v} = \frac{\chi_s}{(\chi_x - \underline{g})(\chi - \overline{g})}\sigma Y.$$

Plugging in equation (48) (which was derived in the baseline model, but the exact same derivation applies in the Coles–Mortensen model), I obtain

$$\Pi = \left(\frac{\chi_x - \overline{g}}{\chi - \overline{g}} + \frac{\chi_x - \overline{g}}{\chi_x - \underline{g}}\frac{\chi_s}{\chi - \overline{g}}\right)\sigma Y.$$

Hence, the aggregate labor share is

$$\mathrm{LS} = 1 - \alpha - \left(\frac{\chi_x - \overline{g}}{\chi - \overline{g}} + \frac{\chi_x - \overline{g}}{\chi_x - \underline{g}}\frac{\chi_s}{\chi - \overline{g}}\right)\sigma.$$

B.3. *Endogenous Vacancy Posting Extension*

I now describe an extension of the baseline model with endogenous vacancy posting (endogenous-λ model). The derivations are brief but detailed. Unless stated otherwise, the environment and notation is exactly as in the baseline model.

Matching

In the baseline model, $\lambda > 0$ is an exogenous parameter that determines the rate at which firms meet workers. I will now interpret λ as a firm-level vacancy rate (i.e., the measure of vacancies per employee). Suppose that firms must pay a cost $c(\lambda, z)$ to post a measure λ of vacancies per employee, which potentially depends on their productivity z. Note that the cost function is allowed to depend directly on the firm type z.

Strategies and Beliefs

As in the baseline model, I restrict attention to Markov strategies that depend only on productivity. The wage schedule, vacancy rate, entry decision, and exist decision are respectively given by $w(z)$, $\lambda(z)$, $e(z)$, and $x(z)$.

Distributions

Define $\widetilde{P}(w)$ to be the wage distribution and $\widetilde{\lambda}(w)$ to be the vacancy rate of firms who pays a wage w. The vacancy-weighted wage distribution $\widetilde{Q}$ and aggregate vacancy rate Λ

are defined as

$$\widetilde{Q}(w) \equiv \frac{1}{\Lambda}\int_0^w \widetilde{\lambda}(w)\,\mathrm{d}\widetilde{P}(w), \tag{54}$$

$$\Lambda \equiv \int_0^\infty \widetilde{\lambda}(w)\,\mathrm{d}\widetilde{P}(w). \tag{55}$$

I denote the employment-weighted and vacancy-weighted productivity distribution as $P(z)$ and $Q(z)$, respectively.

Worker Problem

Denote by U and $W(w)$ the value of being unemployed and the value of being employed at a firm currently paying w, respectively. Equations (3) and (4) in the baseline model become

$$\begin{aligned} rU = b &+ \underbrace{\mu \int e(z)\big|W\big(w(z)\big) - U\big|_+ \,\mathrm{d}\Gamma_0(z)}_{\text{job offers by entering firm}} \\ &+ \underbrace{\Lambda(1-u)\int \big|W\big(w(z)\big) - U\big|_+ \,\mathrm{d}Q(z)}_{\text{job offers by continuing firm}}, \end{aligned} \tag{56}$$

$$\begin{aligned} rW(w) = w &+ \underbrace{\chi \int \big(1 - x(z)\big)\big(W\big(w(z')\big) - W(w)\big)\,\mathrm{d}\Gamma_0\big(z'\big)}_{\text{wage changes due to productivity shocks}} \\ &+ \underbrace{\Lambda(1-u)\int \big|W\big(w(z')\big) - W(w)\big|_+ \,\mathrm{d}Q\big(z'\big)}_{\text{job offers by continuing firm}} \\ &+ \underbrace{\left(\chi \int x(z)\,\mathrm{d}\Gamma_0(z) + \delta\right)\big(U - W(w)\big)}_{\text{job destruction}}. \end{aligned} \tag{57}$$

The only difference with the baseline model is that $\mathrm{d}P(z)$ is replaced by $\mathrm{d}Q(z)$. Hence, the optimal behavior of workers remains as in the baseline model.

Firm Growth

The instantaneous change in employment at a firm of size N_t paying $w_t \geq \underline{w}$ with a vacancy rate λ_t is now given by

$$\mathrm{d}N_t = \widetilde{g}(w_t, \lambda_t) N_t \,\mathrm{d}t,$$

where the employment growth function can be expressed as

$$\widetilde{g}(w,\lambda) = \underbrace{\lambda u + \lambda(1-u)\widetilde{P}(w)}_{\equiv \widetilde{h}(w,\lambda)} - \underbrace{\big(\Lambda(1-u)\big(1 - \widetilde{Q}(w)\big) + \delta\big)}_{\equiv \widetilde{s}(w)}.$$

I define $\widetilde{h}(w, \lambda)$ and $\widetilde{s}(w)$ to be the hiring and separation rates, respectively. The hiring rate can be decomposed as the product of the vacancy rate and the vacancy yield (i.e., measure of hires per vacancy):

$$\widetilde{h}(w, \lambda) = \lambda \times \big(u + (1-u)\widetilde{P}(w)\big).$$

Unlike in the baseline model, firms now have two ways to increase their hiring: posting more vacancies or increasing their wage. The following lemma characterizes the employment growth function.

LEMMA 12: *The function $\widetilde{g}(w, \lambda)$ has the following properties*:

$$\begin{aligned} \widetilde{g}_w(w, \lambda) &= \lambda(1-u)\widetilde{P}'(w) + \widetilde{\lambda}(w)(1-u)\widetilde{P}'(w), \\ \widetilde{g}_\lambda(w, \lambda) &= u + (1-u)\widetilde{P}(w), \\ \lim_{w\to\infty} \widetilde{g}(w, \lambda) &= \lambda - \delta. \end{aligned}$$

Firm Problem

I now characterize the firm problem. In the baseline model, the Cobb–Douglas assumption ensures that the capital share is α for all firms. In the endogenous-λ extension, value-added per worker is $y = zk^\alpha - c(\lambda, z)$. To ensure that both models are completely comparable, I now assume that gross output per worker is exogenously given by $y(z) = zk(z)^\alpha$, where $k(z)$ is given by (15), and that payments to capital represent a fixed share α of value-added. The firm problem is thus given by

$$\begin{aligned} rv(z) = \max_{w\geq b, \lambda\geq 0} \bigg\{ &(1-\alpha)\big(y(z) - c(\lambda, z)\big) - w + v(z)\widetilde{g}(w, \lambda) \\ &+ \chi\bigg(\int v(x)\,\mathrm{d}\Gamma_0(x) - v(z)\bigg)\bigg\} \end{aligned} \tag{58}$$

for all firms who do not exit (i.e., $z \geq \underline{z}$).[23] The first-order condition for vacancy posting is

$$(1-\alpha)c_\lambda\big(\lambda(z), z\big) = \widetilde{g}_\lambda\big(w(z), \lambda(z)\big)v(z). \tag{62}$$

The first-order condition for wages is

$$1 = v(z)\widetilde{g}_w\big(w(z), \lambda(z)\big)$$

[23]The full firm problem can be expressed as a linear complementarity problem. Equations (10), (11), and (12) in the baseline model become:

$$rv(z) = \max_{w\geq b, \lambda\geq 0} \bigg\{(1-\alpha)\big(y(z) - c(\lambda, z)\big) - w + v(z)\widetilde{g}(w, \lambda) + \chi\bigg(\int v(x)\,\mathrm{d}\Gamma_0(x) - v(z)\bigg)\bigg\}, \tag{59}$$

$$rv(z) \geq 0, \tag{60}$$

$$\begin{aligned} 0 = v(z)\bigg(rv(z) - \max_{w\geq b, \lambda\geq 0}\bigg\{ &(1-\alpha)\big(y(z) - c(\lambda, z)\big) - w + v(z)\widetilde{g}(w, \lambda) \\ &+ \chi\bigg(\int v(x)\,\mathrm{d}\Gamma_0(x) - v(z)\bigg)\bigg\}\bigg). \end{aligned} \tag{61}$$

$$= v(z)2\lambda(z)(1-u)\widetilde{P}'(w(z)),$$

where the second equality uses Lemma 12. Using the fact that $P'(z) = \widetilde{P}'(w(z))w'(z)$, we have that

$$w'(z) = v(z)2(1-u)\lambda(z)P'(z).$$

Using the boundary condition $w(\underline{z}) = \underline{w}$, we have that

$$w(z) = \underline{w} + 2(1-u)\int_{\underline{z}}^{z} v(x)\lambda(x)\,\mathrm{d}P(x). \tag{63}$$

Laws of Motion

The laws of motion (21) and (22) in the baseline model become:

$$\dot{u} = \underbrace{(\delta+\chi_x)(1-u)}_{\text{unemployment inflows}} - \underbrace{u\big(\mu_e + \Lambda(1-u)\big)}_{\text{unemployment outflows}}, \tag{64}$$

$$\dot{P}(z) = \underbrace{(1-u)\Lambda P(z)\big(Q(z)-1\big)}_{\text{job-to-job flows}} + \underbrace{\frac{u}{1-u}\mu_e\Gamma(z) + u\Lambda Q(z)}_{\text{employment inflows}}$$

$$- \underbrace{(\delta+\chi_x)P(z)}_{\substack{\text{employment}\\ \text{outflows}}} + \underbrace{\chi_s\big(\Gamma(z) - P(z)\big)}_{\text{productivity shocks}}. \tag{65}$$

Notice that the stationary employment-weighted productivity distribution $P(z)$ is no longer the solution to a quadratic equation, which means that it can no longer be solved analytically.

Functional Form

I assume the following function form for the recruitment cost function:

$$c(\lambda, z) = \frac{\underline{\lambda}^{-\theta}}{(1-\alpha)(1+\theta)}\lambda^{1+\theta}v(z).$$

Under this assumption, I obtain closed-form expression for some equilibrium objects.

LEMMA 13: *Denote the equilibrium vacancy yield (hires per vacancy) as $f(z) \equiv u + (1-u)P(z)$. The equilibrium objects $\lambda(z)$, Λ, $Q(z)$ are given by*

$$\lambda(z) = \underline{\lambda} f(z)^{\frac{1}{\theta}}, \tag{66}$$

$$\Lambda = \underline{\lambda}\frac{\theta}{1+\theta}\frac{1-u^{1+\frac{1}{\theta}}}{1-u}, \tag{67}$$

$$Q(z) = \frac{f(z)^{1+\frac{1}{\theta}} - u^{1+\frac{1}{\theta}}}{1-u^{1+\frac{1}{\theta}}}. \tag{68}$$

PROOF: The policy function $\lambda(z)$ is obtained directly from the first-order condition (62). To obtain an expression for Λ, $Q(z)$, I first define a function $\phi(z) = \int_0^z \frac{\lambda(x)}{\underline{\lambda}} \, \mathrm{d}P(x)$. We have

$$\begin{aligned} \phi(z) &= \int_0^z f(z)^{\frac{1}{\theta}} \, \mathrm{d}P(z) \\ &= \frac{1}{1-u} \int_{f(0)}^{f(z)} x^{\frac{1}{\theta}} \, \mathrm{d}x \\ &= \frac{\theta}{1+\theta} \frac{f(z)^{1+\frac{1}{\theta}} - u^{1+\frac{1}{\theta}}}{1-u}. \end{aligned}$$

Finally, evaluating the definitions (54) and (55) at $w = w(z)$, we have that $\Lambda = \lim_{z\to\infty} \underline{\lambda}\phi(z)$ and $Q(z) = \Lambda^{-1}\underline{\lambda}\phi(z)$. *Q.E.D.*

ASSUMPTION 1: *The rate condition in the baseline model* (*i.e.*, *Assumption* 2) *becomes*

$$\sup_z \lambda(z) < \chi + \delta \quad \iff \quad \underline{\lambda} < \chi + \delta.$$

Defining the equilibrium hiring and separation rates $h(z) \equiv \widetilde{h}(\lambda(z), w(z))$, $s(z) = \widetilde{s}(w(z))$ as well as the vacancy-posting cost $c(z) \equiv c(\lambda(z), z)$, I obtain

$$h(z) = \underline{\lambda} f(z)^{1+\frac{1}{\theta}}, \tag{69}$$

$$s(z) = \Lambda(1-u)\big(1 - Q(z)\big) + \delta, \tag{70}$$

$$c(z) = \frac{1}{(1+\theta)(1-\alpha)} h(z) v(z). \tag{71}$$

Finally, firm-level labor shares are thus given by

$$\mathrm{LS}(z) = \frac{w(z)}{y(z) - \dfrac{1}{(1+\theta)(1-\alpha)} h(z) v(z)}.$$

B.4. *Partially-Segmented Labor Market Extension*

I now solve a stripped-down version of the baseline model with multiple industries. The model is static and only contains two workers and two firms in each industry. Instead of facing search frictions (which in the baseline model limit the firm-level growth rate of employment), firms face an exogenous capacity constraint (i.e., they can hire at most one worker).

The goal of the model is to show that the labor share in an industry is decreasing in its own level of productivity dispersion even when labor markets are not perfectly segmented along industry lines. I first solve the equilibrium under "perfectly-segmented labor market" and then solve the equilibrium under "partially-segmented labor markets."

Perfectly-Segmented Labor Market

There is a continuum of industries $j \in [0, 1]$. In each industry, there are two firms $i \in \{l, h\}$ that differ in productivity and two identical workers. Firm productivity is

$$z_{lj} = 1 - \sigma_j, \qquad z_{hj} = 1 + \sigma_j.$$

The average productivity is normalized to one and the parameter $0 < \sigma_j < 1$ represents productivity dispersion. The average level of productivity dispersion across industries is denoted $\overline{\sigma} \equiv \mathbb{E}\sigma_j$.

Firms compete à la Bertrand by posting wages subject to a capacity constraint (i.e., they can employ at most one worker). Workers observe the wages at both firms and choose to work at the firm with the highest wage. If both firms post the same wage, they each hire one worker. The problem of firm $i \in \{l, h\}$ is

$$\max_{w_i} z_i \min\{N_i, 1\} - w_i N_i,$$

$$\text{s.t.} \quad N_i = \begin{cases} 0 & \text{if } w_i < w_{-i}, \\ 1 & \text{if } w_i = w_{-i}, \\ 2 & \text{if } w_i > w_{-i}, \end{cases}$$

where the subscript $-i$ denotes the competitor of firm i. The outcome of Bertrand competition is well known: both firms offer a wage equal to the productivity of firm l. The following proposition expresses the equilibrium labor share as a function of productivity dispersion.

PROPOSITION 2—Perfectly-segmented labor markets: *The labor share in industry j is*

$$\mathrm{LS}_j = 1 - \sigma_j.$$

PROOF: First, I compute the Nash equilibrium. The best response functions, which express the optimal wage of firm i as a function of the wage offered by firm $-i$ are

$$w_{lj}(w_{hj}) = \begin{cases} w_{hj} & \text{if } w_{hj} \leq z_{lj}, \\ 0 & \text{if } w_{hj} > z_{lj}, \end{cases} \qquad w_{hj}(w_{lj}) = \begin{cases} w_{lj} & \text{if } w_{lj} \leq z_{hj}, \\ 0 & \text{if } w_{lj} > z_{hj}. \end{cases}$$

Therefore, the only Nash equilibrium is $w_{lj} = w_{hj} = z_{lj}$ and the resulting labor allocation is $N_{lj} = N_{hj} = 1$. The resulting labor share in industry j is $\mathrm{LS}_j = \frac{w_{lj}+w_{hj}}{z_{lj}+z_{hj}} = 1 - \sigma_j$. *Q.E.D.*

Importantly, the labor share is strictly decreasing in the level of productivity dispersion σ_j. Note that the emergence of profits in equilibrium is due to the combination of a capacity constraint and productivity dispersion. The capacity constraint has a similar effect to search frictions in the baseline model (i.e., high-productivity firms are not able to absorb all workers).

Partially-Segmented Labor Markets

I now consider the case where there is mobility of workers across industries (i.e., where firms sometimes hire workers located outside of their own industry).

Suppose that, with probability $1-\pi$, the high-productivity firm in industry j competes for workers with the low-productivity firm in another randomly-selected industry j'. Symmetrically, the high-productivity firm in industry j' competes with the low-productivity firm in industry j. With complementary probability π, firms in industry j compete with each other for workers in industry j, as in the perfectly-segmented case.

I interpret the parameter π as the amount of labor market segmentation. If $\pi = 1$, the labor market is perfectly segmented along industry lines: firms within an industry only compete with each other for workers. In contrast, if $\pi < 1$, the labor market is partially-segmented along industry lines: in some cases, firms compete for workers with firms in other industries.

Suppose that firms observe who they are competing with before posting wages. The following proposition expresses the expected equilibrium labor share as a function of own-industry productivity dispersion.

PROPOSITION 3—Partially-segmented labor markets: *The expected labor share in industry j conditional on own-industry productivity dispersion σ_j is*

$$\mathbb{E}(\mathrm{LS}_j\,|\,\sigma_j) = \pi(1-\sigma_j) + (1-\pi)\left(1 - \frac{1}{2}(\sigma_j - \overline{\sigma})\right).$$

PROOF: In industries j that have a segmented labor market, the labor share is exactly as in Proposition 2. In industries j in which the high-productivity firm competes for workers with the low-productivity firm in another industry j', the same logic applies (i.e., the low-productivity firm always posts a wage equal to its productivity and the high-productivity firm always posts a wage equal to the productivity of its competitor). In such industries, the labor share is $\frac{z_{lj}+z_{lj'}}{z_{lj}+z_{hj}} = 1 - \frac{1}{2}(\sigma_j + \sigma_{j'})$. The expected labor share in industry j conditional on own-industry productivity dispersion σ_j is therefore given by $\mathbb{E}(\mathrm{LS}_j\,|\,\sigma_j) = \pi(1-\sigma_j) + (1-\pi)(1-\frac{1}{2}(\sigma_j - \overline{\sigma}))$. *Q.E.D.*

Two remarks are in order. First, industry labor shares are strictly decreasing in their level of productivity dispersion σ_j. Second, the relationship between labor share and own-industry productivity dispersion is weaker (in absolute value) when labor markets are not perfectly segmented along industry lines (i.e., $\pi < 1$).

APPENDIX C: SOLUTION ALGORITHM AND CALIBRATION

C.1. *Solution Algorithm*

I now present a brief but detailed description of the solution algorithm used to solve the baseline model.

Firm Problem

First, I combine the linear complementarity problem representation (LCP) of the firm problem (10), (11), (12) with the expressions for the wage schedule (16), capital per worker (15), and employment growth function $g(z)$ (18). I obtain the following equations:

$$rv(z) \geq (1-\alpha)y(z) - \underline{w} - \int_{\underline{z}}^{z} v(x)g'(x)\,dx + v(z)g(z) + \chi \int v(x)\Gamma_0(dx) - \chi v(z),$$

$$v(z) \geq 0,$$

$$v(z)\Big(rv(z)-\Big((1-\alpha)y(z)-\underline{w}-\int_{\underline{z}}^{z}v(x)g'(x)\,dx+v(z)g(z)$$
$$+\chi\int v(x)\Gamma_0(dx)-\chi v(z)\Big)\Big)=0.$$

To simplify notation, I define the the linear operators $\mathcal{A}, \mathcal{B}, \mathcal{C}$, respectively, defined by the actions

$$\mathcal{A}f(z)=\int_{\underline{z}}^{z}f(x)g'(x)\,dx, \qquad \mathcal{B}f(z)=f(z)g(z),$$
$$\mathcal{C}f(z)=\int v(x)\Gamma_0(dx), \qquad \mathcal{I}f(z)=f(z).$$

The LCP can expressed in terms of the operators

$$rv(z)\geq(1-\alpha)y(z)-\underline{w}-\mathcal{A}v(z)+\mathcal{B}v(z)+\chi\mathcal{C}v(z)-\chi\mathcal{I}v(z),$$
$$v(z)\geq 0,$$
$$v(z)\big(rv(z)-\big((1-\alpha)y(z)-\underline{w}-\mathcal{A}v(z)+\mathcal{B}v(z)+\chi\mathcal{C}v(z)-\chi\mathcal{I}v(z)\big)\big)=0,$$

or more compactly as

$$v(z)\big(\mathcal{M}v(z)+q(z)\big)=0, \qquad \mathcal{M}v(z)+q(z)\geq 0, \quad v(z)\geq 0,$$

where $\mathcal{M}\equiv(r+\chi)\mathcal{I}+\mathcal{A}-\mathcal{B}-\chi\mathcal{C}$ and $q(z)\equiv\underline{w}-(1-\alpha)y(z)$. The threshold $\underline{z}$ does not appear anywhere but can recovered from the solution $v(z)$ as $\underline{z}\equiv\inf\{z: v(z)>0\}$. I now consider a discrete approximation of the system of equations over a grid $\{z_1,\ldots,z_N\}$. Let $v\equiv(v(z_1),\ldots,v(z_N))'$ and $y\equiv(y(z_1),\ldots,y(z_N))'$. I approximate the operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{I}$ by the finite difference method and obtain $N\times N$ matrices A, B, C, I, and M. The resulting system of equations is given by

$$v^T(Mv+q)=0, \qquad Mv+q\geq 0, \quad v\geq 0,$$

which is a plain-vanilla LCP that can be solved with standard routines.

Worker Problem

Define the excess value of working at a firm of productivity z as opposed to being unemployed by $V(z)\equiv W(z)-U$. Evaluating (3), (4), (5) and plugging in the optimal firm decisions (Proposition 2), unemployment rate (23), and employment-weighted distribution (24), I obtain

$$rV(z)=w(z)-b+(\chi-\mu)\int V(z')\,\mathrm{d}\Gamma_0(z')-\lambda(1-u)\int_0^z V(z')\,\mathrm{d}P(z')$$
$$-\lambda(1-u)\overline{P}(z)V(z)-(\delta+\chi)V(z),$$
$$V(z)\geq 0.$$

In operator form, the system is

$$\begin{aligned} rV(z) &= w(z) - b + (\chi-\mu)\mathcal{C}V(z) - \lambda(1-u)\mathcal{D}V(z) - \lambda(1-u)\mathcal{E}V(z) \\ &\quad - (\delta+\chi)\mathcal{I}V(z), \\ V(z) &\geq 0, \end{aligned}$$

where $\mathcal{C}, \mathcal{I}$ were defined earlier and

$$\mathcal{D}f(z) = \int_0^z f(z)\,\mathrm{d}P(z), \qquad \mathcal{E}f(z) = \big(1-P(z)\big)f(z).$$

More compactly, we have

$$V(z)\big(\mathcal{M}V(z)+q(z)\big) = 0, \qquad \mathcal{M}V(z)+q(z) \geq 0, \quad V(z) \geq 0,$$

where

$$\mathcal{M} = (r+\chi+\delta)\mathcal{I} - (\chi-\mu)\mathcal{C} + \lambda(1-u)(\mathcal{D}+\mathcal{E}), \qquad q(z) = b - w(z)$$

I now consider a discrete approximation of the system of equations over a grid $\{z_1, \dots, z_N\}$. I approximate the operators $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{I}$ by the finite difference method and obtain $N \times N$ matrices C, D, E, I, and M. The resulting system of equations is given by

$$V^T(MV+q) = 0, \qquad MV+q \geq 0, \quad V \geq 0,$$

which again is a LCP that can be solved with standard routines. I recover the reservation wage as

$$\underline{w} = b + (\mu-\chi)CV.$$

Algorithm

The equilibrium allocation (unemployment rate, employment-weighted distribution) can be computed in closed form (3) but depends on the equilibrium thresholds $(\underline{z}, \underline{w})$. The algorithms consists of iterating over the worker and firm problem until convergence. The algorithm is initiated (step $b=0$) with a guess $(\underline{z}^{(0)}, \underline{w}^{(0)})$. Then a typical step b consists of the following operations:

1. Using $\underline{z}^{(b-1)}$, $\underline{w}^{(b-1)}$, solve the firm problem. Collect the wage $\underline{w}^{(b)}$ and the threshold $\underline{z}^{(b)}$.
2. Using $\underline{z}^{(b)}$ and $\underline{w}^{(b)}$, solve the worker problem. Collect the reservation wage $\underline{w}^{(b)}$.
3. If $\max\{|\underline{z}^{(b)} - \underline{z}^{(b-1)}|, |\underline{w}^{(b)} - \underline{w}^{(b-1)}|\} < 10^{-6}$, stop. Otherwise, continue to step $b+1$.

Solving the Firm Size Distribution

In Table V, I estimate a regression of firm labor share LS on size N in the data. To compute the model-implied regression coefficient, I need to solve for the joint-distribution of firm size N and productivity z. To do so, I use the Pareto extrapolation method developed in Gouin-Bonenfant and Toda (2022). Since the method applies to the discrete-time model, I discretize the model over short time intervals of one week.

C.2. *Solution Algorithm (Endogenous-λ Extension)*

I now present the solution algorithm used to solve the endogenous-λ model. I pay particular attention to where it differs from the baseline solution algorithm (see Appendix C.1).

Firm Problem

Define $c(z) \equiv c(\lambda(z), z)$. The firm problem can be solved exactly as in the baseline model, the only differences being the operator $\mathcal{A}$, which is now defined as

$$\mathcal{A}f(z) = \int_{\underline{z}}^{z} 2\lambda(x)(1-u)f(x)\,\mathrm{d}x,$$

and the function $q(z)$, which is now defined as

$$q(z) = \underline{w} + c(z) - (1-\alpha)y(z).$$

Worker Problem

As explained in Section B.3, the worker problem is nearly identical as in the baseline model, which can be discretized and solved numerically.

Unemployment and Employment-Weighted Productivity Distribution

I solve for the stationary employment-weighted productivity distribution $P(z)$ and unemployment rate u by iterating over their laws of motion (i.e., equations (64) and (65)) until convergence. Let $P_n^{(b)}$, $\lambda_n^{(b)}$ be the discrete approximation of $P(z_n)^{(b)}$, $\lambda(z_n)^{(b)}$ at iteration b, where z_n is a point on the productivity grid. Given $\lambda_n^{(b)}$, $u^{(b)}$, $P_n^{(b)}$, I use equations (54) and (55) combined with numerical integration to approximate $\{Q_1^{(b)}, Q_2^{(b)}, \dots\}$ and $\Lambda^{(b)}$.

The updating equations, where (b) denotes the iteration, are

$$\begin{aligned}
u^{(b+1)} &= u^{(b)} + \Delta\big[(\delta+\chi_x)\big(1-u^{(b)}\big) - u^{(b)}\big(\mu_e + \Lambda^{(b)}\big(1-u^{(b)}\big)\big)\big],\\
P_n^{(b+1)} &= P_n^{(b)} + \Delta\Bigg[\big(1-u^{(b+1)}\big)\Lambda^{(b)}P_n^{(b)}\big(Q_n^{(b)}-1\big) + \frac{u^{(b+1)}}{1-u^{(b+1)}}\mu_e\Gamma_n + u^{(b+1)}\Lambda^{(b)}Q_n^{(b)}\\
&\qquad - (\delta+\chi_x)P_n^{(b)} + \chi_s\big(\Gamma_n - P_n^{(b)}\big)\Bigg],\\
\lambda_n^{(b+1)} &= \frac{1}{c}\Big(\big(u^{(b+1)} + \big(1-u^{(b+1)}\big)P_n^{(b+1)}\big)\Big)^{\frac{1}{\vartheta}}v_n,
\end{aligned}$$

where I set the time step Δ to 1/12 (i.e., monthly). I iterate until $\max\{|u^{(b+1)} - u^{(b)}|, |P_n^{(b+1)} - P_n^{(b)}|\} < 10^{-6}$.

Algorithm

Similar to the algorithm for the baseline model, the algorithm consists of iterating over (i) the firm problem, (ii) the worker problem, and (iii) the unemployment and employment-weighted productivity distribution. The algorithm is initiated (step $b = 0$) with a guess $(\underline{z}^{(0)}, \underline{w}^{(0)})$ and $\lambda^{(0)} = (\lambda(z_1)^{(0)}, \dots, \lambda(z_N)^{(0)})'$.

TABLE C.I

CHOICE OF PRODUCTIVITY DISTRIBUTION.

Distribution	Dispersion	Skewness
Data	1.547	−0.046
Gamma	1.547	−0.027
Pareto	1.547	+0.465

Note: Dispersion is defined as the interdecile range of log labor productivity $\log P90 - \log P10$; skewness is defined as the Kelley skewness index, which is defined as $(\log P90 - 2\log P50 + \log P10)/(\log P90 - \log P10)$. The notation $P10$, $P50$, and $P90$ represent the 10th, 50th, and 90th percentiles of the distribution of log labor productivity (net of 3-digit NAICS industry and year fixed effects).

C.3. *Calibration*

Choice of Productivity Distribution

I now discuss why I use the Gamma distribution to model the distribution of labor productivity across firms, rather than a Pareto distribution considered in Proposition 6. In short, the reason is that the Pareto distribution implies too much skewness.

The first row of Table C.I contains the dispersion and skewness of log labor productivity in the data (as well as the precise definition of "dispersion" and "skewness"). The dispersion is 1.547 (which is the empirical target in the calibration exercise) and the skewness is small and slightly negative (−0.046). The second row contains the dispersion and skewness of log labor productivity in the baseline model calibrated using a Gamma distribution (see Section 3.1 for the calibration strategy). Notice the the (non-targeted) skewness coefficient is small and negative (−0.027), as in the data. The last row contains the dispersion and skewness in the baseline model calibrated with a Pareto distribution (see Section 5.4 for a discussion of the calibration strategy). The key observation is that the skewness is large and positive (+0.465). Hence, the Gamma distribution appears to be a more appropriate functional form for the empirical distribution of labor productivity.

Model-Implied Moments (for Calibration)

The unemployment rate is obtained directly from (23) while the the interdecile range of labor productivity is computed numerically. I now derive the formulas used to compute the five other moments in the model.

To compute the job creation and destruction rates by continuing firms analytically, it will be useful to obtain an implicit equation for the productivity threshold z_0 such that $g(z_0) = 0$. Straightforward algebra implies that

$$g(z_0) = 0 \quad \Longleftrightarrow \quad P(z_0) = \frac{\lambda(1-u) + \delta - \lambda u}{2\lambda(1-u)}.$$

The job creation rate by continuing firms is defined as $\int_{z_0}^{\infty} g(z)\,\mathrm{d}P(z)$, and can be expressed as

$$\int_{z_0}^{\infty} g(z)\,\mathrm{d}P(z) = \big(1 - P(z_0)\big)\big(\lambda u - \lambda(1-u) - \delta\big) + \lambda(1-u)\big(1 - P(z_0)^2\big).$$

Similarly, the job destruction rate by continuing firms is

$$-\int_{\underline{z}}^{z_0} g(z)\,\mathrm{d}P(z) = -P(z_0)\big(\lambda u - \lambda(1-u) - \delta\big) - \lambda(1-u)P(z_0)^2.$$

The measure of jobs created by entering firms over a period $[t, t+1)$ is $u\mu_e$ and the measure of jobs created by exiting firms is $(1-u)\chi_x$, where μ_e, χ_x are defined in (20). Therefore, the job creation (destruction) by entrants (exiters) are $\frac{u}{1-u}\mu_e$ and χ_x, respectively. As discussed in Section 3.1, the continuous-time growth rates g_c are transformed into discrete-time growth rates g_d using the formula $g_d = e^{g_c} - 1$.

Finally, to derive the autocorrelation of log labor productivity, I first approximate the law of motion for labor productivity y_t as

$$\log y_{t+1} \approx \begin{cases} \log y_t & \text{with prob. } e^{-\chi_s}, \\ x & \text{with prob. } 1 - e^{-\chi_s}, \end{cases}$$

where x is drawn from the stationary distribution of $\log y_t$ and χ_s is defined in (20). Denoting the stationary mean of $\log y_t$ as μ, notice that the autocorrelation of $\log y_t$ is

$$\begin{aligned} &\frac{\mathbb{E}(\log y_{t+1} - \mu)(\log y_t - \mu)}{\mathbb{E}(\log y_t - \mu)^2} \\ &\quad = \frac{e^{-\chi_s}\mathbb{E}(\log y_t - \mu)^2 + \left(1 - e^{-\chi_s}\right)\mathbb{E}(\log y_t - \mu)(x - \mu)}{\mathbb{E}(\log y_t - \mu)^2} = e^{\chi_s}. \end{aligned}$$

The last step uses the fact that $\mathbb{E}(\log y_t - \mu)(x - \mu) = 0$ since x is independent of $\log y_t$.

I now derive expressions for the pass-through and separation elasticities (see Section 3.2 for definitions and context). To be consistent with the empirical evidence, the pass-through elasticity is computed using the OLS formula (i.e., $\text{cov}(\log w(z), \log y(z))/\text{var}(\log y(z))$). Similarly, the separation elasticity is given by $\text{cov}(\log S(z), \log w(z))/\text{var}(\log w(z))$, where $S(z) = 1 - e^{-\lambda(1-u)P(z)-\delta}$.[24] In both cases, z is drawn from the employment-weighted productivity distribution $P(z)$.

Jacobian Matrix

Let $\Lambda(\theta) = (\Lambda_1(\theta), \dots, \Lambda_m(\theta), \dots \Lambda_7(\theta))$ be the vector of model-implied moments, where $\theta = (\theta_1, \dots, \theta_p, \dots, \theta_6)$ is the vector of model parameters. To quantify the sensitivity of the model-implied moments to the choice of parameters, I compute the numerical derivative

$$\left.\frac{\partial \Lambda_m(\theta)}{\partial \theta_p}\right|_{\theta=\theta^*},$$

for all $(m, p) \in \{1, \dots, 7\} \times \{1, \dots, 6\}$, where θ^* is the vector of calibrated parameters. Table C.II reports the results.

C.4. *Alternative Calibrations*

Table C.III reports the model fit in the benchmark models and extensions.

[24]Recall that the flow of separations is given by exogenous lay-offs, which occur at Poisson rate δ, and endogenous separations, which occur at Poisson rate $\lambda(1-u)P(z)$

TABLE C.II

SENSITIVITY OF MODEL-IMPLIED MOMENTS TO PARAMETERS.

Moment	Parameter					
	λ	χ	μ	η	b	δ
Unemployment rate	0.117	−0.878	0.332	−0.001	0.496	1.810
Autocorrelation of log labor prod.	0.102	−1.170	0.192	−0.001	0.208	0.030
Interdecile range of log labor prod.	−1.445	5.953	−2.723	−0.204	−2.951	−0.421
Job creation (continuers)	0.289	−0.054	0.020	0.000	0.030	−0.403
Job creation (entrants)	0.026	−0.241	0.161	0.000	0.140	0.608
Job destruction (continuers)	0.184	0.174	−0.066	0.000	−0.098	0.187
Job destruction (exiters)	0.128	−0.458	0.242	−0.001	0.263	0.037

APPENDIX D: DATA

D.1. *Variables Construction*

I now describe the methodology used to construct the main firm-level variables (i.e., value-added, employment, average wage, and capital stock). I start with four variables constructed by Statistics Canada that are based on corporate tax return line items: gross profits, worker compensation, tangible capital assets, and intangible capital assets. The last two variables (tangible capital assets and intangible capital assets) represent book values of assets net of accumulated depreciation. First, I construct value added as

$$\text{value added} \equiv \text{gross profits} + \text{worker compensation}.$$

This approach is consistent with the income approach to measuring GDP which, in the corporate sector, sums the income which accrues to firm owners (gross operating income) and the income which accrues to workers (worker compensation). Labor share and labor productivity are defined as

$$\text{labor share} \equiv \text{worker compensation}/\text{value added},$$

$$\text{labor productivity} \equiv \text{value added}/\text{employment},$$

TABLE C.III

TARGETED MOMENTS.

Moment	Data	Model				
		Baseline	CM	BM	E-λ	Pareto
Unemployment rate	0.071	0.071	0.071	0.071	0.075	0.069
Autocorrelation of log labor prod.	0.810	0.806	0.806	–	0.803	0.806
Interdecile range of log labor prod.	1.547	1.547	1.547	1.547	1.544	1.547
Job creation (continuers)	0.061	0.055	0.055	–	0.050	0.055
Job creation (entrants)	0.019	0.022	0.022	–	0.024	0.022
Job destruction (continuers)	0.067	0.062	0.062	–	0.059	0.062
Job destruction (exiters)	0.016	0.016	0.016	–	0.015	0.015
EU transition rate	0.047	–	–	0.047	–	–

Note: "CM" refers to Coles–Mortensen; "BM" refers to Burdett–Mortensen; "Pareto" refers to the baseline model with a Pareto distribution; "E-λ" refers to the endogenous-λ extension of the baseline model.

TABLE D.I

SUMMARY STATISTICS (2000–2015).

Variable	Observations	Mean	Std. deviation
Employment	3,084,182	38.6	571.1
Value-added	3,084,182	2480.0	45,579.6
Capital stock	3,084,182	2733.4	90,548.0
Compensation per worker	3,084,182	37.8	27.6
Labor productivity	3,084,182	52.8	185.2

Note: All variables except employment are in thousands of 2002 Canadian dollars using the CPI deflator.

where employment is obtained by averaging the monthly number of employees throughout the year. The capital stock is the sum of tangible and intangible capital assets measured at book value net of accumulated depreciation.

Finally, I winsorize the labor share at the 0.1% level in the upper tail (i.e., labor share values above the 99.9 percentile are replaced by the value of the 99.9 percentile, which is approximately 20). I also remove from the main sample firm-year observations that either have negative value-added or missing values in employment, value-added, tangible capital assets, intangible capital assets or industry code. Table D.I contains summary statistics.

D.2. *Sample Validation*

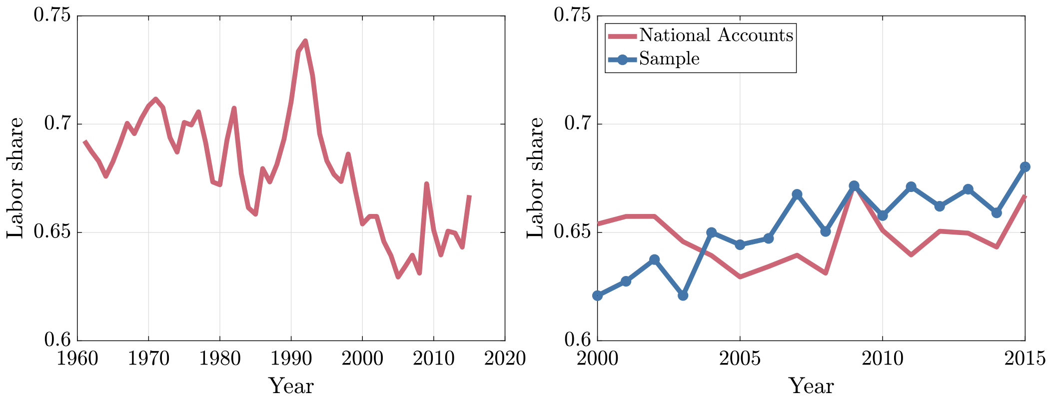
I now compare the aggregate labor share in the NALMF versus in the National Accounts. I use data from Statistics Canada (Table 380-0063) to compute the corporate sector labor share. As is standard, I assume that the components of income that are ambiguous (i.e., taxes net of subsidies and net mixed income) have a labor share equal to the aggregate labor share, which implies the following approximation:

$$\begin{aligned}&\text{aggregate labor share}\\&\quad\approx \frac{\text{worker compensation}}{\text{worker compensation} + \text{gross operating surplus}}.\end{aligned}$$

Figure D.1 plots the labor share in the National Accounts and in the main sample. A few remarks are in order. First, the Canadian labor share in the National Accounts has sustained a large decline over the course of the 1990s and early-2000s but has then somewhat recovered over the course of the 2005–2015 period. Second, the aggregate labor share in the main sample has a similar level and trend as the one in the National Account over the period where both data sets are available. Overall, the level and dynamics of the labor share in the NALMF is consistent with the aggregate data, except for the fact that it is too low in the first 4 years of the sample.

D.3. *Statistics by Labor Productivity Deciles*

Within each 2-digit NAICS industry-year bin, I sort firms by labor productivity (i.e., value-added per worker) and bin firms into deciles. Within each decile, I compute labor productivity, capital-output ratio, value-added share, and labor share. Finally, I average



(a) Aggregate labor share

(b) Aggregate labor share

FIGURE D.1.—Labor share in the National Accounts and in the main sample.

over all industries within a year by applying value-added weights and compute a simple average of each variable over 2000–2015. Table D.II contains the resulting data. Note that labor productivity and capital-output ratio are expressed in relative term (i.e., as a ratio of the aggregate).

D.4. *Comparison Between Compustat and Census Data*

Table 3 in Barth, Bryson, Davis, and Freeman (2016) contains a measure of productivity dispersion in the U.S. The authors use data from the Census Bureau's Economic Census files and compute the variance of log revenue per worker in the cross-section of establishments. The data is based on quinquennial censuses of establishments, so they report values every 5 years. In Compustat, I compute the variance of log revenue per worker every 5 years from 1982 to 2007. I present the results in Table D.III. Overall, the levels are very comparable and both series show an increase in productivity dispersion despite differences in concept. The unit of observation in Compustat (as in the NALMF data) is a firm, while in Barth et al. (2016) it is an establishment. The main difference in trend appears to be from 2002 to 2007, where the increase in productivity dispersion is noticeably higher in Compustat.

TABLE D.II

MOMENTS BY LABOR PRODUCTIVITY DECILES.

	Labor productivity deciles									
Variable	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Labor productivity	0.164	0.354	0.434	0.518	0.596	0.692	0.783	0.973	1.278	2.889
Capital-output ratio	1.800	0.811	0.694	0.621	0.691	0.694	0.749	0.799	0.917	1.259
Value-added share	0.015	0.028	0.036	0.044	0.055	0.069	0.082	0.097	0.149	0.425
Labor share	1.578	1.126	0.954	0.911	0.858	0.802	0.761	0.742	0.667	0.439

TABLE D.III

PRODUCTIVITY DISPERSION IN COMPUSTAT VERSUS CENSUS DATA.

Years	1982	1987	1992	1997	2002	2007
Barth et al. (2016)	0.965	0.949	1.020	1.113	1.126	1.265
Compustat	0.950	1.024	1.048	1.185	1.162	1.474

D.5. *Industry-Level Data Set*

To construct the industry-level data set, I use the main sample (described in Appendix D.1) and sort firms by labor productivity within industry-year bins and assign to each firm a productivity quintile. An industry is defined according to 3-digit NAICS definitions. Within each industry-year bin, I compute the labor share and value-added shares. I restrict the sample to industry-year observations that have at least 100 firms in every year. The result is a balanced panel data set covering 69 industries over the 2000–2015 period. Table D.IV reports summary statistics.

APPENDIX E: QUANTIFYING THE THEORY: ROBUSTNESS CHECKS

E.1. *Assessing Cross-Sectional Moments*

Table E.I reproduces and extends the results presented in Table V. In specification (3), I instrument labor productivity with its 1-year lag. The coefficient on labor productivity goes up from −0.31 to −0.25. In specification (6), I regress labor share on labor productivity with the addition of the capital-output ratio as a control. The coefficient on labor productivity remains mostly unchanged and the coefficient on capital-output ratio is negative −0.025, as expected. Also, adding the capital-output output ratio increases the R^2 slightly, from 0.27 to 0.28. In specification (7), I use total worker compensation as an alternative measure of firm size (i.e., instead of employment). As with employment, the coefficient on size is close to zero and the R^2 is low.

The main takeaway is that differences in labor productivity across firms explain a large fraction of the variance in labor shares (the year and industry fixed effects explain less than 3% of the variance) and that, conditional on labor productivity, size, and capital-output ratio do not provide much additional predictive power. In the U.S. manufacturing sector, Kehrig and Vincent (2021) also find that differences in labor shares across firms are mostly explained by differences in labor productivity, rather than differences in capital intensity.

TABLE D.IV

SUMMARY STATISTICS (CROSS-INDUSTRY DATA SET).

Variable	Observations	Mean	Std. deviation
Labor share	1104	0.672	0.135
Productivity dispersion	1104	1.917	0.473

Note: Productivity dispersion is defined as the interdecile range of log labor productivity.

TABLE E.I
ASSESSING CROSS-SECTIONAL MOMENT (ADDITIONAL SPECIFICATIONS).

Labor share (log LS)	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Value-added ($\log Y$)	−0.112 (0.001)						
Labor productivity ($\log y$)		−0.313 (0.002)	−0.248 (0.002)	−0.313 (0.002)		−0.320 (0.002)	
Employment ($\log N$)				0.009 (0.001)	−0.001 (0.001)		
Capital-output ($\log K/Y$)						−0.025 (0.000)	
Payroll ($\log wN$)							0.033 (0.000)
Industry fixed effects	✓	✓	✓	✓	✓	✓	✓
Year fixed effects	✓	✓	✓	✓	✓	✓	✓
2SLS			✓				
Sample size	3,084,182	3,084,182	2,480,615	3,084,182	3,084,182	3,084,182	3,084,182
R^2	0.120	0.272	0.256	0.273	0.036	0.282	0.042

Note: Standard errors in parentheses are clustered at the firm level.

E.2. *Do High-Wage Firms Have a Lower Labor Share?*

In Section 3.2, I test (and find support for) a number of model predictions including (i) high-productivity firms have a lower labor share and (ii) high-productivity firms pay higher wages. Combining these two model predictions, one would also expect that that high-wage firms have a low labor share. However, when I estimate the cross-sectional relationship between labor share and wage in the data (where wage is defined as worker compensation per employee), I instead find that high-wage firms tend to have a slightly *higher* labor share.[25] In particular, the regression coefficient of log labor share on log wage (with year and industry fixed effects exactly as in Table V) is 0.11 in the data compared to −1.06 in the model (see the first row of Table E.II).

To understand why the model fails to match this particular moment, consider the following accounting decomposition. For any two random variables u and v, let $\beta_{u,v}$ denote the regression coefficient of the log of u on the log of v and let σ_u^2 denote the variance of the log of u.[26] Using the fact that the log of labor share is equal to the log of wage minus

TABLE E.II
WAGES AND LABOR SHARE IN THE CROSS-SECTION OF FIRMS.

Moment	Data	Model
$\beta_{\mathrm{LS},w}$	0.11	−1.06
$\beta_{w,y}$	0.69	0.39
σ_y^2/σ_w^2	1.29	5.22

[25] Kehrig and Vincent (2021) also estimate a small (but negative) relationship between wage and labor share across establishments in the U.S. manufacturing sector (see their Figure VI).

[26] The regression coefficient is defined as $\beta_{u,v} \equiv \mathrm{cov}(\log u, \log v)/\mathrm{var}(\log v)$.

the log of labor productivity (i.e., $\log \text{LS} = \log w - \log y$), we have that

$$\beta_{\text{LS},w} = 1 - \beta_{w,y} \times \left(\sigma_y^2/\sigma_w^2\right).$$

Table E.II contains each component of the decomposition, both in the data (where all variables are net of year and industry fixed effects) and in the model. The regression coefficient of log wage on log productivity $\beta_{w,y}$ is 0.69 in the data, which is fairly close to the model's prediction of 0.39. In contrast, the ratio of variances σ_y^2/σ_w^2 is 1.29 in the data, while it is 5.22 in the model. Hence, the main reason why the model fails to match the empirical evidence on the relationship between productivity and wage $\beta_{\text{LS},w}$ is because it underpredicts the variance of wages relative to the variance of labor productivity.

In the model, a single firm-level state variable (i.e., productivity z) determines the equilibrium wage that firms offer. In reality, there are other factors besides productivity that generate dispersion in average wages across firms, even within narrowly defined industry (i.e., the presence of a union, compensating wage differentials, etc.). To understand how such factors would bias the regression coefficient $\beta_{\text{LS},w}$, consider the following stylized model extension. Suppose that wages are given by

$$w(z,\xi) = \xi w(z),$$

where $w(z)$ is the policy function in the baseline model and ξ is an exogenous wedge that is orthogonal to productivity z, has a cross-sectional mean of one, and has a log variance of σ_ξ^2. Under these assumptions, the regression coefficient of log labor share on log wages $\beta_{\text{LS}(z,\xi),w(z,\xi)}$ is given by

$$\begin{aligned}\beta_{\text{LS}(z,\xi),w(z,\xi)} &\equiv 1 - \beta_{w(z,\xi),y(z)} \times \left(\sigma_{y(z)}^2/\sigma_{w(z,\xi)}^2\right) \\ &= 1 - \beta_{w(z),y(z)} \times \left(\sigma_{y(z)}^2/\sigma_{w(z)}^2\right) \times \frac{\sigma_{w(z)}^2}{\sigma_{w(z)}^2 + \sigma_\xi^2}.\end{aligned}$$

A couple of remarks are in order. First, the presence of wedges ξ does not affect the regression coefficient of log wages on log labor productivity (i.e., $\beta_{w(z,\xi),y(z)} = \beta_{w(z),y(z)}$). This is a standard result: classical measurement errors in the independent variable does not bias the regression coefficient. Second, the ratio of variances has a multiplicative bias equal to $\sigma_w^2/(\sigma_w^2 + \sigma_\xi^2)$. For instance, as the the variance of the wedge σ_ξ^2 goes to infinity, the regression coefficient $\beta_{LS(z,\xi),w(z,\xi)}$ goes to one.

The key takeaway is that the presence of wedges ξ in the wage equation (i.e., unmodeled determinants of firm-level wages) biases the regression coefficient $\beta_{\text{LS},w}$ (even its sign), but does not affect the estimated relationship between labor share and productivity $\beta_{\text{LS},y}$. To match the relationship between labor share and wage in the data, one would need a richer model that generates more wage dispersion between firms.

E.3. *Moments by Labor Productivity Decile: Robustness Checks and Extensions*

I now describe various robustness checks and extensions of Figure 2 in Section 3.2, which plots the labor share by labor productivity decile.

Industry and Size Effects

In Figure E.1a, I group firms in the data into broad industry categories (i.e., goods producing, trade services, professional services, and other services). Similarly, Figure E.1b groups firms in the data into size groups (50–20, 20–100, 100–500, and 500+ employees). The main empirical findings hold within each of these groups: (i) there is a negative relationship between firm-level labor share and labor productivity and (ii) a large fraction of firms have a labor share above $1-\alpha$.

Measurement Error in Value-Added

To alleviate the concern that measurement error in value-added could generate a spurious negative relationship between labor share and labor productivity (see footnote 13), Figure E.1c reproduces Figure 2 by sorting firms according to their 1-year lagged labor productivity decile. The results are similar, the main difference being that the labor share of firms in the bottom decile of labor productivity is lower.

Benchmark Models and Endogenous-λ Extension

Figure E.1d plots the relationship between labor share and labor productivity in the calibrated benchmark models (i.e., Burdett–Mortensen and Coles–Mortensen). There are two takeaways. First, the Burdett–Mortensen and Coles–Mortensen models fail to match the high labor share (i.e., above $1-\alpha$) of low-productivity firms. Second, the labor share in the endogenous-λ model extension is very similar to the labor share in the baseline model. The main difference is that the labor share of high-productivity firms is higher (i.e., high-productivity firms exert less monopsony power).

Pareto Distribution

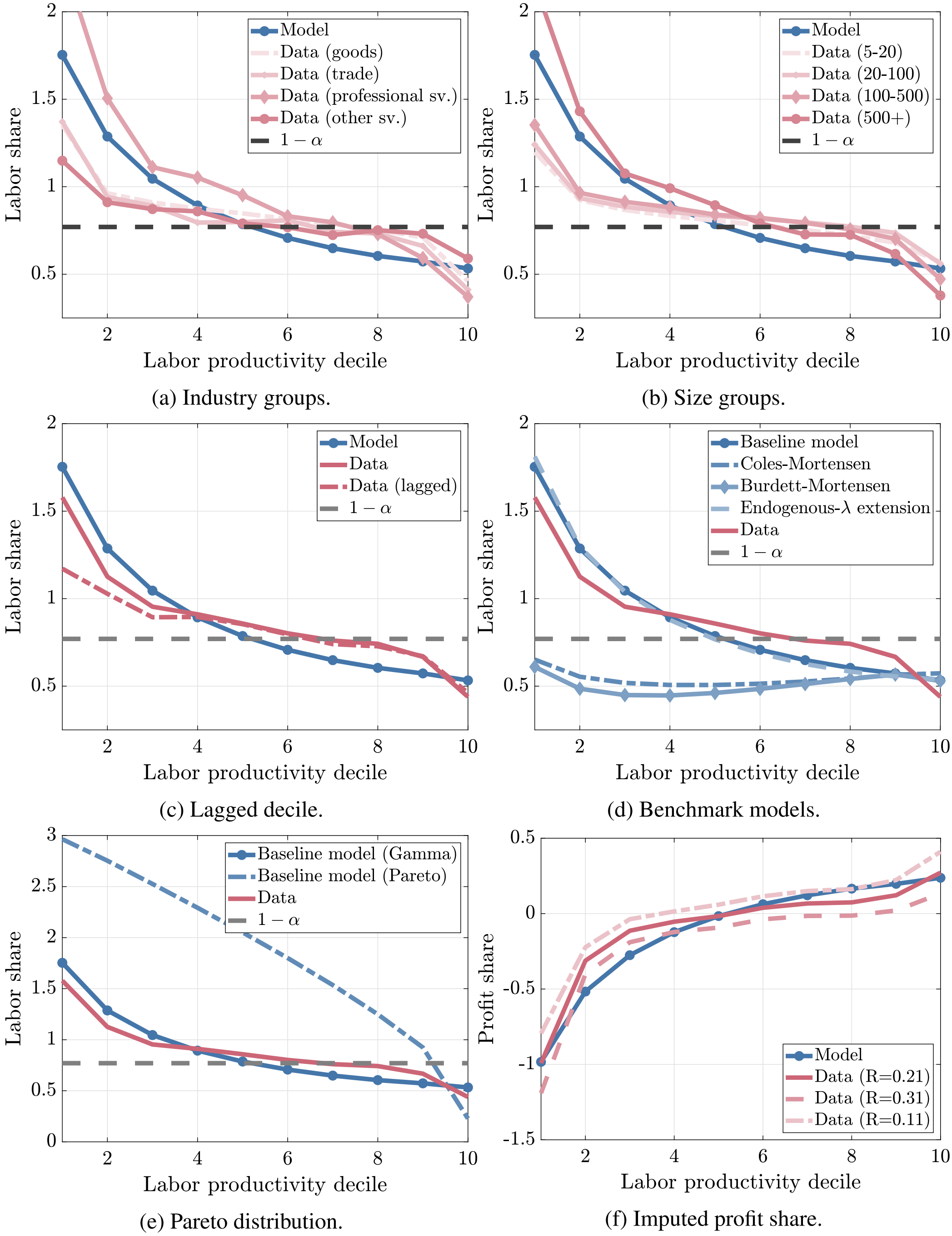
Figure E.1e plots the relationship between labor share and labor productivity in the baseline model calibrated with a Pareto distribution. The empirical fit is poor. In particular, the labor share in the bottom 9 deciles of labor productivity are much higher than in data. Moreover, the labor share in the top decile that is much too low.

Imputed Profit Share

The empirical relationship between labor productivity and labor share could in principle be explained by the fact that firms differ in their capital share α, even within a narrowly-defined industry. Kehrig and Vincent (2021) find that, in the US manufacturing sector, differences in capital per worker explain a negligible fraction of the variation in labor shares across firms. I now conduct a similar exercise in my data. First, notice that if firms face the same user cost of capital R, the statistic that is informative about the relative capital share $R \times K/Y$ is the capital-output ratio K/Y. I impute the profit share Π/Y as

$$\underbrace{\Pi/Y}_{\text{profit share}} = \underbrace{1-\mathrm{LS}}_{\text{nonlabor share}} - \underbrace{R \times K/Y}_{\text{capital share}}, \tag{72}$$

where both the nonlabor share $1-\mathrm{LS}$ and the capital output ratio K/Y are directly observable. This approach has a long tradition in economics dating back to Jorgenson (1963) and has been recently used by Barkai (2020) using U.S. aggregate data. Figure E.1f provides a comparison of the (imputed) profit shares in the data and in the model. I find that the profit share increases with labor productivity. Both in the model and in the data, the bottom 5 deciles have a negative profit share while the top 5 have a positive one. Moreover, the qualitative results are not sensitive to the particular value of the user cost that I use. As a robustness check, I plot the profit shares in the data implied by a user cost of 0.11 and 0.31 instead of 0.21 as in the baseline model. The shape of the relationship between labor productivity and profit share remains mostly unchanged.



(a) Industry groups.

(b) Size groups.

(c) Lagged decile.

(d) Benchmark models.

(e) Pareto distribution.

(f) Imputed profit share.

FIGURE E.1.—Moments by labor productivity decile: robustness checks and extensions.

TABLE E.III

CROSS-COUNTRY REGRESSIONS.

LS	(1)	(2)	(3)
σ	−0.077	−0.201	−0.119
	(0.026)	(0.035)	(0.085)
Year FE	✓	✓	
Industry FE	✓	✓	✓
Country FE		✓	
N	154	154	28
R^2	0.245	0.858	0.143

Note: Robust standard errors. "LS" denotes labor share; "σ" denotes productivity dispersion (i.e., the interdecile range of log labor productivity). For specifications (3), the windows are 2001–2006, 2006–2011, and 2010–2015. For specifications (4), the window is 2001–2011.

E.4. *Cross-Country Regressions*

In Table E.III, I estimate specifications (38), (39), and (40). In all specifications, I obtain negative coefficients ranging from −0.077 (specification 38) to −0.220 (specification 40 with 10-year differences). Despite the small sample size, I detect a clear negative relationship between labor share and productivity dispersion in all specifications.

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